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B.Sc. (Mathematics)

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FUZZY ALGEBRA

Author:
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Phone: 0120-4078900 • Fax: 0120-4078999
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• Website: www.vikaspublishing.com • Email: helpline@vikaspublishing.com

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SYLLABI-BOOK MAPPING TABLE

Fuzzy Algebra

Syllabi	Mapping in Book
BLOCK I: FUZZY SETS AND OPERATIONS ON FUZZY SETS **UNIT-1** Fuzzy Sets - Basic Types - Basic Concepts - Cuts - Additional Properties of - Cuts - Extension Principle for Fuzzy Sets. **UNIT- 2** Operations on Fuzzy Sets - Types of Operations - Fuzzy Complements - Fuzzy Union and Intersections. **UNIT- 3** Combinations of Operations - Fuzzy Arithmetic - Fuzzy Numbers. **UNIT- 4** Arithmetic Operations on Intervals - Arithmetic Operations on Fuzzy Numbers.	**Unit 1:** Fuzzy Sets **(Pages 1-14);** **Unit 2:** Operations on Fuzzy Sets **(Pages 15-23);** **Unit 3:** Combinations and Fuzzy Numbers **(Pages 24-35);** **Unit 4:** Arithmetic Operations **(Pages 36-51)**
BLOCK II: FUZZY RELATIONS AND FUZZY MEASURES **UNIT- 5** Fuzzy Relations - Binary Fuzzy Relations - Fuzzy Equivalence and Similarity Relations - Fuzzy Compatibility Relations. **UNIT- 6** Fuzzy Ordering Relations - Fuzzy Morphisms. **UNIT- 7** Fuzzy Measures - Belief and Plausibility Measures - Probability Measures - Problems. **UNIT- 8** Possibility Measures - Necessity Measures - Relationship among Classes of Fuzzy Measures.	**Unit 5:** Ordering Relations on Fuzzy Sets **(Pages 52-63);** **Unit 6:** Fuzzy Morphism **(Pages 64-77);** **Unit 7:** Fuzzy Measures **(Pages 78-90);** **Unit 8:** Possibility Measures and Necessity Measures **(Pages 91-97)**
BLOCK III: UNCERTAINTY AND MEASURES OF DISSONANCE **UNIT- 9** Types of Uncertainty - Measures of Fuzziness - Problems. **UNIT- 10** Classical Measures of Uncertainty - Hartley Information - Shannon Entropy - Boltzmann Entropy. **UNIT- 11** Measures of Dissonance - Body of Evidence - Consonant Body of Evidence - Problems.	**Unit 9:** Types of Uncertainty **(Pages 98-105);** **Unit 10:** Classical Measures of Uncertainty **(Pages 106-115);** **Unit 11:** Measures of Dissonance **(Pages 116-123)**

BLOCK IV: MEASURES OF CONFUSION, UNCERTAINTY AND INFORMATION

UNIT- 12

Measures of Confusion - Entropy like Measures - Problems.

UNIT- 13

Measures of Non-Specificity - Uncertainty - Problems.

UNIT- 14

Uncertain and Information - Syntactic, Semantic, Pragmatic - Problems.

CONTENTS

INTRODUCTION

NOTES

Fuzzy algebra is a branch of mathematics which deals with the fuzzy set theory and fuzzy logic. It started in 1965 after the publication of Lotfi Asker Zadeh's seminal work "*Fuzzy Sets*". Indeed, Zadeh elaborated Cantor's binary set theory to a gradual model by presenting degrees of belonging and relationship. In an instant, this extension was applied to almost all the domains of modern mathematics. Consequently, some new disciplines such as fuzzy geometry, fuzzy databases, fuzzy algebraic structures, fuzzy arithmetic, fuzzy differential calculus, fuzzy topology, fuzzy relational calculus, and fuzzy decision making were appeared.

In mathematics, fuzzy sets (uncertain sets) are somewhat like sets whose elements have degrees of membership. Salii (1965), defined a more general kind of structure called an L-relation, which he studied in an abstract algebraic context. Fuzzy relations, which are now used throughout fuzzy mathematics and have applications in areas such as linguistics (De Cock, Bodenhofer & Kerre 2000), decision-making (Kuzmin 1982), and clustering (Bezdek 1978), are special cases of L-relations when L is the unit interval $[0, 1]$.

In mathematics, a fuzzy subset A of a set X is a function $A: X \to L$, where L is the interval $[0, 1]$. This function is also called a membership function. A membership function is a generalization of an indicator function (also called a characteristic function) of a subset defined for $L = \{0, 1\}$. More generally, one can use any complete lattice L in a definition of a fuzzy subset A.

Fuzzy logic is a form of many-valued logic in which the truth value of variables may be any real number between 0 and 1 both inclusive. It is employed to handle the concept of partial truth, where the truth value may range between completely true and completely false. By contrast, in Boolean logic, the truth values of variables may only be the integer values 0 or 1. Fuzzy logic is based on the observation that people make decisions based on imprecise and non-numerical information. Fuzzy models or sets are mathematical means of representing vagueness and imprecise information (hence the term fuzzy). These models have the capability of recognising, representing, manipulating, interpreting, and utilising data and information that are vague and lack certainty. Fuzzy logic has been applied to many fields, from control theory to artificial intelligence.

This book, *Fuzzy Algebra*, is divided into four blocks, which are further subdivided into fourteen units. The topics discussed include fuzzy sets, a-cuts, extension principle for fuzzy sets, operations on fuzzy sets, fuzzy complements, fuzzy union and intersections, combinations of operations, fuzzy arithmetic, fuzzy numbers, arithmetic operations on fuzzy numbers, fuzzy relations, binary fuzzy relations, fuzzy equivalence and similarity relations, fuzzy compatibility relations, fuzzy ordering relations, fuzzy morphisms, fuzzy measures, probability measures, possibility measures, necessity measures, types of uncertainty, measures of fuzziness, Hartley information, Shannon entropy, Boltzmann entropy, measures of dissonance, body of evidence, measures of confusion, entropy like measures, measures of non-specificity, uncertain information, and syntactic, semantic and pragmatic information.

NOTES

The book follows the Self-Instructional Mode (SIM) wherein each unit begins with an 'Introduction' to the topic. The 'Objectives' are then outlined before going on to the presentation of the detailed content in a simple and structured format. 'Check Your Progress' questions are provided at regular intervals to test the student's understanding of the subject. 'Answers to Check Your Progress Questions', a 'Summary', a list of 'Key Words', and a set of 'Self-Assessment Questions and Exercises' are provided at the end of each unit for effective recapitulation.

BLOCK - I
FUZZY SETS AND OPERATIONS ON FUZZY SETS

NOTES

UNIT 1 FUZZY SETS

Structure

1.0 INTRODUCTION

In mathematics, fuzzy sets (also known as uncertain sets) are somewhat like sets whose elements have degrees of membership. Fuzzy sets were introduced independently by Lotfi A. Zadeh and Dieter Klaua [de] in 1965 as an extension of the classical notion of set. At the same time, Salii (1965) defined a more general kind of structure called an *L*-relation, which he studied in an abstract algebraic context. Fuzzy relations, which are now used throughout fuzzy mathematics and have applications in areas, such as linguistics (De Cock, Bodenhofer & Kerre 2000), decision-making (Kuzmin 1982), and clustering (Bezdek 1978), are special cases of *L*-relations when *L* is the unit interval [0, 1].

In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition — an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set. A membership function valued in the real unit interval [0, 1]. Fuzzy sets generalize classical sets, since the indicator functions (also known as characteristic functions) of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1. In fuzzy set theory, classical bivalent sets are usually called crisp sets. The fuzzy set theory can be used in a wide range of domains in which information is incomplete or imprecise, such as bioinformatics.

NOTES

Type-2 fuzzy sets and systems generalize standard Type-1 fuzzy sets and systems so that more uncertainty can be handled. From the beginning of fuzzy sets, criticism was made about the fact that the membership function of a Type-1 fuzzy set has no uncertainty associated with it, something that seems to contradict the word fuzzy, since that word has the connotation of much uncertainty. A Type-2 fuzzy set lets us incorporate uncertainty about the membership function into fuzzy set theory, and is a way to address the above criticism of Type-1 fuzzy sets head-on. And, if there is no uncertainty, then a Type-2 fuzzy set reduces to a Type-1 fuzzy set, which is analogous to probability reducing to determinism when unpredictability vanishes.

Type-1 fuzzy systems are working with a fixed membership function, while in Type-2 fuzzy systems the membership function is fluctuating. A fuzzy set determines how input values are converted into fuzzy variables.

L. A. Zadeh proposed the concept of fuzzy sets in a study published in 1965 (Zadeh, 1965). The fuzzy sets theory proposes working with linguistic variables, representing hazy concepts, and dealing with ambiguous boundaries. In this approach, fuzzy sets have evolved as a new technique of dealing with uncertainty.

The fuzzy multigroup notion in this theory is an algebraic structure of a fuzzy multiset that generalizes both the classical and fuzzy groups. In fact, fuzzy multigroup is a fuzzy multiset application to the basic notion of classical groups. The concept of upper and lower α-cuts of fuzzy multigroups are introduced, and some of their features are investigated in this study. The concept of fuzzy multigroup α-cuts serves as a link between classical groups and fuzzy multigroups. We prove that α-cuts of a fuzzy multigroup are subgroups of a larger group within certain bounds.

One of the most fundamental concepts in fuzzy set theory is the extension principle. It establishes a general way for applying sharp mathematical notions to fuzzy quantities, such as real algebraic operations on fuzzy numbers. These operations are computationally efficient interval analysis generalizations.

In this unit, you will study about the fuzzy set, basic types of fuzzy sets, basic concepts of fuzzy sets, additional properties of α-cuts and extension principle for fuzzy sets.

1.1 OBJECTIVES

After going through this unit, you will be able to:

- Define the fuzzy sets
- Explain the basic types of fuzzy sets
- Analysis the basic concepts of fuzzy sets
- Elaborate on the additional properties of α-cuts
- Discuss about the extension principle for fuzzy sets

1.2 FUZZY SET

NOTES

Fuzzy sets (also known as uncertain sets) are somewhat like sets whose elements have degrees of membership. Fuzzy sets were introduced independently by Lotfi A. Zadeh & Dieter Klaua in 1965 as an extension of the classical notion of set. At the same time, Salii (1965) defined a more general kind of structure called an *L*-relation, which he studied in an abstract algebraic context. Fuzzy relations, which are now used throughout fuzzy mathematics and have applications in areas, such as linguistics (De Cock, Bodenhofer and Kerre 2000), decision-making (Kuzmin 1982) and clustering (Bezdek 1978) are special cases of *L*-relations when L is the unit interval [0, 1].

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Definition

μ_A A fuzzy set is a pair (U,m) where U is a set (often required to be non-empty) and m: $U \rightarrow [0,1]$ a membership function. The reference set U (sometimes denoted by Ω or X) is called universe of discourse, and for each $x \in U$, the value $m(x)$ is called the grade of membership of x in (U, m). The function $m = \mu A$ is called the membership function of the fuzzy set $A = (U, m)$

For a finite set $U = \{x_1, \ldots, x_n\}$, the fuzzy set (U, m) is often denoted by $\{m(x_1)/x_1, \ldots, m(x_n)/x_n\}$.

Let $x \in U$. Then x is

- Not included in the fuzzy set (U, m) if $m(x) = 0$ (no member),
- Fully included if $m(x) = 1$ full member),
- Partially included if $0 < m(x) < 1$ (fuzzy member).

The crisp set of all fuzzy sets on a universe U is denoted with $SF(U)$ (or sometimes just $F(U)$).

Crisp Sets Related to a Fuzzy Set

For any fuzzy set $A = (U, m)$ and $\alpha \in [0,1]$ the following crisp sets are defined:

NOTES

- $A^{\geq\alpha} = A_\alpha = \{x \in U \mid m(x) \geq \alpha\}$ is called its α-cut (also known as α-level set)
- $A^{>\alpha} = A'_\alpha = \{x \in U \mid m(x) > \alpha\}$ is called its strong α-cut (also known as strong α-level set)
- $S(A) = \mathrm{Supp}(A) = A^{>0} = \{x \in U \mid m(x) > 0\}$ is called its support
- $C(A) = \mathrm{Core}(A) = A^{=1} = \{x \in U \mid m(x) = 1\}$ is called its core (or sometimes kernel (A)).

1.3 BASIC TYPES OF FUZZY SETS

The basic definition of a fuzzy set was presented, as well as the initial set theoretic procedures. The membership space was supposed to be a real-number space, the membership functions were crisp functions, and the operations were simply dual logic or Boolean algebra operations.

They could be about the definition of a fuzzy set or about operations on fuzzy sets. Different structures can be placed on the membership space and different assumptions can be made about the membership function when defining a fuzzy set. These add-ons will be dealt with.

The set theoretic intersection, which is modelled by the min-operator, corresponds to the logical 'AND' the logical 'OR' the union, and the max-operator were all assumed to have the same type of relationship. Alternative and additional definitions for terms like intersection and union, as well as their interpretation as 'AND' and 'OR' and mathematical models, might be developed by departing from the well-established systems of dual logic and Boolean algebra.

We've only looked at fuzzy sets with well-defined membership functions or degrees of membership so far. It's debatable whether humans, for example, have or can have a clear mental concept of how membership functions. As a result, Zadeh [1973] proposed the concept of a fuzzy set whose membership function is also a fuzzy set. If Type-1 fuzzy sets, such as those discussed thus far, are considered, a Type-2 fuzzy set can be described as follows.

Definition

A Type-2 fuzzy set has membership values that are Type-1 fuzzy sets on the range [0, 1].

NOTES

For Type-2 fuzzy sets, the operations intersection, union, and complement are no longer sufficient. Until now, we've covered the extension principle, which will come in this situation. Similarly, to how Type-2 fuzzy sets were established, it may be argued that there is no clear reason why Type-2 fuzzy set membership functions should be crisp. The notion of Type-m fuzzy sets is a natural extension of existing Type-2 fuzzy sets.

A Type-m Fuzzy Set on *X*

A Type-m fuzzy set in *X* has membership values of Type-m - 1, $m > 1$ fuzzy sets on the range [0, 1].

In practise, such Type-m fuzzy sets for large m (even for $m \geq 3$) are tough to deal with, and measuring or visualising them will be extremely difficult, if not impossible. As a result, we won't even attempt to define the standard operations on them. Other attempts to introduce vagueness beyond the fuzziness of conventional Type-1 fuzzy sets have been made. Norwich and Turksen's 'Stochastic Fuzzy Model' [1981, 1984] is one example. Those writers were primarily concerned with membership function measurement and scale level. They view a fuzzy set as a family of random variables whose density functions are estimated by that stochasticity [Norwich and Turksen 1984]. Hirota (1981) also takes into account fuzziness.

A Probabilistic Set *A* on *X*

μ_A A defining function μ_A defines a probabilistic set A on X.

$$\mu_A : X \times \Omega \ni (x, \omega) \to \mu_A(x, \omega) \in \Omega_C$$

$\mu_A(x, \cdot)$ where is the B, B_C measurable function for each fixed $x \in X$.

One of the main advantages of the notion of probabilistic sets in modelling fuzzy and stochastic features of a system is asserted to be the applicability of moment analysis, that is, the possibility of computing moments, such as expectation and variance. The difference between the appearance of fuzzy sets and probabilistic sets [Hirota 1981]. Of course, the mathematical properties of probabilistic sets differ from those of fuzzy sets, and so do the mathematical models for intersection, union and so on.

An L-fuzzy set [Goguen 1967; De Luca and Termini 1972] is a more general definition of a fuzzy set than that given in definition. An L-fuzzy set's membership function maps to a partially ordered set L. The fuzzy set-in definition is a special L-fuzzy set since the interval [0, 1] is a poset (partially ordered set).

NOTES

Atanassov and Stoeva [Atanassov and Stoeva 1983; Atanassov 1986], who proposed a generalisation of the notion of fuzzy sets-intuitionistic fuzzy sets-and Pawlak [Pawlak 1982], who created the theory of rough sets, where grades of membership are described by a concept of approximation, made further attempts to describe imprecise and uncertain data using other forms of fuzzy sets. Given an underlying set X of objects, an Intuitionistic Fuzzy Set (IFS) is a set of ordered triples,

$$A = \{(x, \mu_A(x), \nu_A(x)) | x \in X\}$$

Ordinary fuzzy sets over X may be viewed as special intuitionistic fuzzy sets with the non-membership function $\nu_A(x) = 1 - \mu_A(x)$. In the same way as fuzzy sets are defined by mapping membership functions into a partially ordered set *L*, intuitionistic *L*-fuzzy sets are defined by mapping membership functions into a partially ordered set *L* [Atanassov and Stoeva 1984].

1.4 BASIC CONCEPTS OF FUZZY SETS

L. A. Zadeh proposed the concept of fuzzy sets in a study published in 1965 (Zadeh, 1965). The fuzzy sets theory proposes working with linguistic variables, representing hazy concepts and dealing with ambiguous boundaries. In this approach, fuzzy sets have evolved as a new technique of dealing with uncertainty.

L-Fuzzy Sets

Sometimes, more general variants of the notion of fuzzy set are used, with membership functions taking values in a (fixed or variable) algebra or structure *L* of a given kind; usually it is required that *L* be at least a poset or lattice. These are usually called *L*-fuzzy sets, to distinguish them from those valued over the unit interval. The usual membership functions with values in [0, 1] are then called [0, 1] valued membership functions. These kinds of generalizations were first considered in 1967 by Joseph Goguen, who was a student of Zadeh. A classical corollary may be indicating truth and membership values by (f, t) instead of (0, 1).

An extension of fuzzy sets has been provided by Atanassov and Baruah. An Intuitionistic Fuzzy Set (IFS) is characterized by two functions:

1. $\mu_A(x)$ - Degree of membership of *x*
2. Degree of non-membership of *x* with functions $\mu_A, \nu_A : U \mapsto [0,1]$

   and $\forall x \in U : \mu_A(x) + \nu_A(x) \leq 1$ this resembles a situation like some person denoted by *x* voting

   - For a proposal A: $(\mu_A(x) = 1, \nu_A(x) = 0)$,

- Against it: $(\mu_A(x) = 0, \nu_A(x) = 1)$.
- Or abstain from voting: $(\mu_A(x) = \nu_A(x) = 0)$.

After all, we have a percentage of approvals, a percentage of denials, and a percentage of abstentions.

For this situation, special 'Intuitive Fuzzy' negators, t- and s-norms can be defined with $D^* = \{(\alpha, \beta) \in [0,1]^2 : \alpha + \beta = 1\}$ and by combining both functions to $(\mu_A, \nu_A) : U \to D^*$ this situation resembles a special kind of L-fuzzy sets.

Once more, this has been expanded by defining Picture Fuzzy Sets (PFS) as follows: A PFS is characterized by three functions mapping U to [0, 1]:

'Degree of Positive Membership' 'Degree of Neutral Membership' and 'Degree of Negative Membership' respectively and additional condition, μ_A, η_A, ν_A.

$$\forall x \in U : \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1$$

This expands the voting sample above by an additional possibility of refusal of voting.

with $D^* = \{(\alpha, \beta, \gamma) \in [0,1]^3 : \alpha + \beta + \gamma = 1\}$

and special 'Picture Fuzzy' negators, t- and s-norms this resembles just another type of L-fuzzy sets.

Neutrosophic Fuzzy Sets

The concept of Intuitionistic Fuzzy Set (IFS) has been extended into two major models. The two extensions of IFS are neutrosophic fuzzy sets and Pythagorean fuzzy sets.

Neutrosophic fuzzy sets were introduced by Smarandache in 1998. Like IFS, neutrosophic fuzzy sets have the previous two functions: one for membership $\mu_A(x)$ and another for non-membership $v_A(x)$. The major difference is that neutrosophic fuzzy sets have one more function: for indeterminate $i_A(x)$. The major difference is that neutrosophic fuzzy sets have one more function: for indeterminate $i_A(x)$ value can be particularly useful when one cannot be very confident on the membership or non-membership values for item x. In summary, neutrosophic fuzzy sets are associated with the following functions:

1. $\mu_A(x)$ - Degree of membership of x
2. $v_A(x)$ – Degree of non-membership of x
3. $i_A(x)$ – Degree of indeterminate value of x

Pythagorean Fuzzy Sets

The other extension of Intuitionistic Fuzzy Set (IFS) is what is known as Pythagorean fuzzy sets. Pythagorean fuzzy sets are more flexible than IFS.

NOTES

IFS are based on the constraint $\mu_A(x) + \nu_A(x) \leq 1$, which can be considered as too restrictive in some occasions. This is why Yager proposed the concept of Pythagorean fuzzy sets. Such sets satisfy the constraint $\mu_A(x)^2 + \nu_A(x)^2 \leq 1$, which is reminiscent of the Pythagorean Theorem. Pythagorean fuzzy sets can be applicable to real life applications in which the previous condition of $\mu_A(x)^2 + \nu_A(x)^2 \leq 1$ may be suitable in more domains.

1.5 ADDITIONAL PROPERTIES OF Á-CUTS

The fuzzy multigroup notion, in theory, is an algebraic structure of a fuzzy multiset that generalizes both the classical and fuzzy groups. In fact, fuzzy multigroup is a fuzzy multiset application to the basic notion of classical groups. The concept of upper and lower α-cuts of fuzzy multigroups are introduced, and some of their features are investigated in this study. The concept of fuzzy multigroup α-cuts serves as a link between classical groups and fuzzy multigroups. We prove that α-cuts of a fuzzy multigroup are subgroups of a larger group within certain bounds.

Properties of α-Cuts I

The α-cuts of any fuzzy set can be used to characterise it. As a result, α-cuts are critical for the application of fuzzy sets.

Theorem 1.1

Let $\mu \in \mathcal{F}(X),\ \alpha \in [0,1]$ *and* $\beta \in [0,1]$.

(*a*) $[\mu]_0 = X$,

(*b*) $\alpha < \beta \Longrightarrow [\mu]_\alpha \supseteq [\mu]_\beta$,

(*c*) $\bigcap_{\alpha:\alpha<\beta} [\mu]_\alpha = [\mu]_\beta$.

Properties of α-Cuts II

Theorem (Representation Theorem)

Let, $\mu \in \mathcal{F}(X)$.

Then,

$$\chi_{[\mu]_\alpha}(x) = \begin{cases} 1, & \text{if } x \in [\mu]_\alpha \\ 0, & \text{otherwise.} \end{cases}$$

NOTES

$$\mu(x) = \sup_{\alpha \in [0,1]} \left\{ \min(\alpha, \chi_{[\mu]_\alpha}(x)) \right\}$$

$L \subseteq [0, 1]$ So, fuzzy set can be obtained as upper envelope of its α-cuts.

Simply draw α-cuts parallel to horizontal axis in height of α.It is advised that in applications, a finite subset $L \subseteq [0, 1]$ of relevant degrees of membership be chosen. They need to be semantically distinct. Fix level sets of fuzzy sets to characterise only certain levels, in other words.

1.6 EXTENSION PRINCIPLE FOR FUZZY SETS

If the input, x, is crisp, then the resulting output, y, is crisp in a mapping supplied by the general function f: $y = f(x)$.

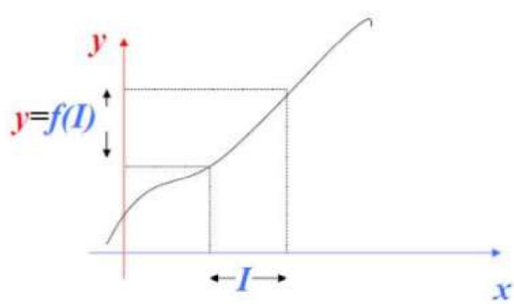

Fig. 1.1 *Creating a Clean Collection of Maps*

We can extend the domain of a function on fuzzy sets using a method established by Zadeh [1975]. As a result, a common point-to-point mapping of a function $f(.)$ is generalised to a mapping between fuzzy sets.

- Let A, B be two fuzzy sets defined in the X, Y discourse universe.
- Let f: $X \rightarrow Y$ be a nonfuzzy transformation function between worlds X and Y.
- The crisp function, as we call it,

  f: $X \rightarrow Y$

When it's expanded to act on fuzzy sets defined on X and Y, it becomes fuzzified.

NOTES

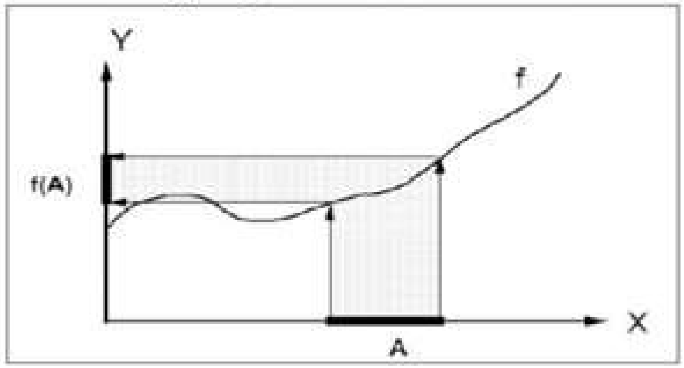

Fig. 1.2 *Mapping Conventional Sets*

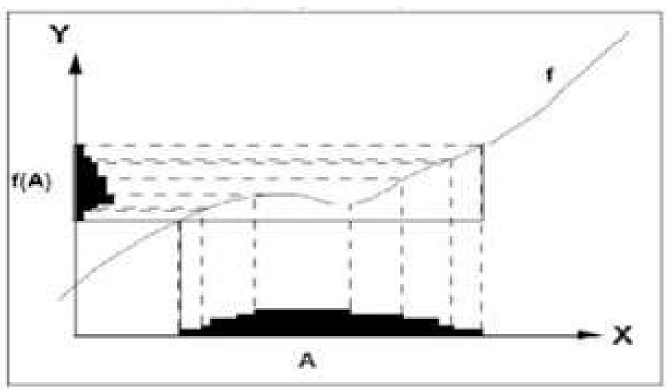

Fig. 1.3 *Mapping Fuzzy Sets*

Vertical Representation

Until now, fuzzy sets have been described by their characteristic/membership function and degree of membership $\mu(x)$ assigned to each element $x \in X$.

That is the vertical representation of the matching fuzzy set, for example, a verbal statement, such as 'Around *m*'.

$$\mu_{m,d}(x) = \begin{cases} 1 - \left|\frac{m-x}{d}\right|, & \text{if } m-d \le x \le m+d \\ 0, & \text{otherwise,} \end{cases}$$

or about halfway between 'B and C'

$$\mu_{a,b,c,d}(x) = \begin{cases} \frac{x-a}{b-a}, & \text{if } a \le x < b \\ 1, & \text{if } b \le x \le c \\ \frac{x-d}{c-d}, & \text{if } c < x \le d \\ 0, & \text{if } x < a \text{ or } x > d. \end{cases}$$

Horizontal Representation Horizontal representation is frequently used is as follows:

Human expert lists components of X that fulfil vague concept of fuzzy set with degree e" α for all membership degrees belonging to chosen subset of [0, 1]. This is the α-cuts' horizontal depiction of fuzzy sets.

Let, $\mu \in \mathcal{F}(X)$ and $\alpha \in [0, 1]$.

Then the sets is,

$$[\mu]_\alpha = \{x \in X \mid \mu(x) \geq \alpha\}, \quad [\mu]_{\underline{\alpha}} = \{x \in X \mid \mu(x) > \alpha\}$$

are known as the α-cut and rigorous α-cut of μ.

Check Your Progress

1. Explain the theory of fuzzy sets?
2. State the basic types of fuzzy sets.
3. Write the additional properties of α-cuts.
4. Define vertical representation.
5. What do you understand by horizontal representation?

1.7 ANSWERS CHECK YOUR PROGRESS

1. A fuzzy set is a pair (U, m) where U is a set (often required to be non-empty) and m: $U \rightarrow [0, 1]$ a membership function. The reference set U (sometimes denoted by Ω or X) is called universe of discourse, and for each $x \in U$, the value $m(x)$ is called the grade of membership of x in (U, m). The function $m = \mu_A$ is called the membership function of the fuzzy set A = (U, m)
2. L-Fuzzy sets: The general variants of the notion of fuzzy set are used, with membership functions taking values in a (fixed or variable) algebra or structure L of a given kind; usually it is required that L be at least a poset or lattice. These are usually called L-fuzzy sets.

   Neutrosophic Fuzzy Sets: The concept of Intuitionistic Fuzzy Set (IFS) has been extended into two major models. The two extensions of IFS are neutrosophic fuzzy sets and Pythagorean fuzzy sets. Neutrosophic fuzzy sets were introduced by Smarandache in 1998.

   Pythagorean Fuzzy Sets: The other extension of Intuitionistic Fuzzy Set (IFS) is what is known as Pythagorean fuzzy sets. Pythagorean fuzzy sets are more flexible than IFSs.

NOTES

3. Properties of α-Cuts I

   The α-cuts of any fuzzy set can be used to characterise it. As a result, α-cuts are critical for the application of fuzzy sets.

   Properties of α-Cuts II

   It is advised that in applications, a finite subset $L \subseteq [0, 1]$ of relevant degrees of membership be chosen. They need to be semantically distinct. Fix level sets of fuzzy sets to characterise only certain levels.

4. Vertical Representation: Fuzzy sets have been described by their characteristic/membership function and degree of membership $\mu(x)$ assigned to each element $x \in X$. That is the vertical representation of the matching fuzzy set, for example, a verbal statement, such as 'Around m.'

5. Horizontal Representation: Human expert lists components of X that fulfil vague concept of fuzzy set with degree $\geq \alpha$ for all membership degrees belonging to chosen subset of [0, 1].

1.8 SUMMARY

- Fuzzy sets are somewhat like sets whose elements have degrees of membership. Fuzzy sets were introduced independently by Lotfi A. Zadeh and Dieter Klaua in 1965 as an extension of the classical notion of set.
- Salii (1965) defined a more general kind of structure called an L-relation.
- In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition-an element either belongs or does not belong to the set.
- The fuzzy set theory can be used in a wide range of domains in which information is incomplete or imprecise, such as bioinformatics.
- A fuzzy set is a pair (U, m) where U is a set (often required to be non-empty) and m: $U \rightarrow [0, 1]$ a membership function.
- The reference set U (sometimes denoted by Ω or X) is called universe of discourse, and for each $x \in U$, the value $m(x)$ is called the grade of membership of x in (U, m). The function $m = \mu_A$ is called the membership function of the fuzzy set $A = (U, m)$.
- The membership space was supposed to be a real-number space, the membership functions were crisp functions, and the operations were simply dual logic or Boolean algebra operations.
- A Type-2 fuzzy set has membership values that are Type-1 fuzzy sets on the range [0, 1].
- A Type-m fuzzy set in X has membership values of Type-m - 1, $m > 1$ fuzzy set on the range [0, 1].

- One of the main advantages of the notion of probabilistic sets in modelling fuzzy and stochastic features of a system is asserted to be the applicability of moment analysis that is the possibility of computing moments, such as expectation and variance.

NOTES

- The more general variants of the notion of fuzzy set are used, with membership functions taking values in a (fixed or variable) algebra or structure *L* of a given kind; usually it is required that *L* be at least a poset or lattice. These are usually called *L*-fuzzy sets
- The concept of Intuitionistic Fuzzy Set (IFS) has been extended into two major models. The two extensions of IFS are neutrosophic fuzzy sets and Pythagorean fuzzy sets.

  Neutrosophic fuzzy sets were introduced by Smarandache in 1998.
- The α-cuts of any fuzzy set can be used to characterise it. As a result, α-cuts are critical for the application of fuzzy sets.
- It is advised that in applications, a finite subset $L \subseteq [0, 1]$ of relevant degrees of membership be chosen. They need to be semantically distinct. Fix level sets of fuzzy sets to characterise only certain levels, in other words.
- If the input, x is crisp, then the resulting output, y is crisp in a mapping supplied by the general function f: $y = f(x)$.
- The crisp function is,

  $f: X \rightarrow Y$

  When it's expanded to act on fuzzy sets defined on *X* and *Y*, it becomes fuzzified.
- Fuzzy sets have been described by their characteristic/membership function and degree of membership $\mu(x)$ assigned to each element $x \in X$. that is the vertical representation of the matching fuzzy set, for example, a verbal statement, such as 'Around *m*'.

1.9 KEY WORDS

- **Crisp sets:** In fuzzy set theory, classical bivalent sets are usually called crisp sets. The fuzzy set theory can be used in a wide range of domains in which information is incomplete or imprecise, such as bioinformatics.
- **Fuzzy sets:** A fuzzy set is a pair (U, m) where U is a set and m: $U \rightarrow [0, 1]$ a membership function. The reference set U is called universe of discourse, and for each $x \in U$, the value $m(x)$ is called the grade of membership of x in (U, m). The function $m = \mu_A$ is called the membership function of the fuzzy set $A = (U, m)$.
- **Pythagorean fuzzy sets:** The other extension of Intuitionistic Fuzzy Set (IFS) is what is known as Pythagorean fuzzy sets. Pythagorean fuzzy sets are more flexible than IFSs. IFSs are based on the constraint

  $\mu_A(x) + v_A(x) \leq 1$

NOTES

- **Vertical representation:** Fuzzy sets have been described by their characteristic/membership function and degree of membership $\mu(x)$ assigned to each element $x \in X$ that is the vertical representation of the matching fuzzy set, for example, a verbal statement, such as 'Around m'.
- **Horizontal representation:** Human expert lists components of X that fulfil vague concept of fuzzy set with degree $\geq \alpha$ for all membership degrees belonging to chosen subset of [0, 1] this is the α-cuts horizontal depiction of fuzzy sets.

1.10 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Describe the term fuzzy set.
2. What are the basic types of fuzzy sets?
3. What do you understand by the additional properties of α-cuts?
4. Differentiate between the vertical representation and horizontal representation.

Long-Answer Questions

1. Explain about the fuzzy sets with the help of suitable examples.
2. Elaborate on the significance of crisp set in fuzzy set giving appropriate examples.
3. Discuss the basic types of fuzzy sets giving examples of each type.
4. Explain about the additional properties of α-cuts with the help of suitable examples.
5. Briefly discuss about the extension principle for fuzzy sets giving examples of each type.

1.11 FURTHER READINGS

Klir, George J. and Bo Yuan. 1995. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. New Jersey: Prentice Hall Inc.

Klir, George J. and Tina A. Folger. 2009. *Fuzzy Sets, Uncertainty and Information*. New Delhi: Prentice Hall of India.

Zimmermann, Hans-Jürgen. 1991. *Fuzzy Set Theory—And Its Applications*. New Delhi: Allied Publishers Private Limited.

UNIT 2 OPERATIONS ON FUZZY SETS

NOTES

Structure

2.0 INTRODUCTION

In mathematics a fuzzy set operation is an operation on fuzzy sets. These operations are generalization of crisp set operations. There is more than one possible generalization. The most widely used operations are called standard fuzzy set operations. There are three operations: fuzzy complements, fuzzy intersections and fuzzy unions.

A fuzzy set is a mathematical representation of ambiguous qualitative or quantitative data obtained using natural language. The model is based on a generalisation of classical set and characteristic function notions. Fuzzy sets are similar to sets whose constituents have different degrees of membership in mathematics. The membership of items in a set is evaluated in binary terms according to the bivalent condition in classical set theory-an element either belongs to the set or does not belong to the set. Fuzzy set theory, allows for a gradual assessment of the membership of elements in a set, which is characterised using a membership function having a value in the real unit interval [0, 1]. Because the indicator functions of classical sets are special instances of the membership functions of fuzzy sets, if the latter only accept values 0 or 1, fuzzy sets generalise classical sets. Crisps sets are the name given to classical bivalent sets in fuzzy set theory. The fuzzy set theory can be used to a wide range of fields, including bioinformatics, when information is incomplete or inaccurate.

The goal of this material is to show how the mathematical structure of the concept of Fuzzy Sets works. The concept of the characteristic function for a set is used to achieve this generalisation.

In this unit, you will study about the fuzzy set operations, fuzzy complement, fuzzy unions and fuzzy intersection.

NOTES

2.1 OBJECTIVES

After going through this unit, you will be able to:

- Discuss about the types of fuzzy set operations
- Describe about the fuzzy complement
- Understand the basic concept of fuzzy unions
- Explain fuzzy intersection

2.2 FUZZY SET OPERATIONS

A fuzzy set operation is an operation on fuzzy sets. These operations are generalization of crisp set operations. There is more than one possible generalization. The most widely used operations are called standard fuzzy set operations. There are three operations:

- Fuzzy Complements
- Fuzzy Intersections
- Fuzzy Unions

A fuzzy set is a mathematical representation of ambiguous qualitative or quantitative data obtained using natural language. The model is based on a generalisation of classical set and characteristic function notions. Fuzzy sets are similar to sets whose constituents have different degrees of membership in mathematics. The membership of items in a set is evaluated in binary terms according to the bivalent condition in classical set theory-an element either belongs to the set or does not belong to the set. Fuzzy set theory, allows for a gradual assessment of the membership of elements in a set, which is characterised using a membership function having a value in the real unit interval [0, 1]. Because the indicator functions of classical sets are special instances of the membership functions of fuzzy sets, if the latter only accept values 0 or 1, fuzzy sets generalise classical sets. Crisps sets are the name given to classical bivalent sets in fuzzy set theory. The fuzzy set theory can be used to a wide range of fields, including bioinformatics, when information is incomplete or inaccurate.

Types of Operations

Standard Fuzzy Set Operations

Let A and B be fuzzy sets that A, B $\subseteq$ U, u is any element (e.g., value) in the U universe: $u \in$ U.

Standard Complement

$$\mu_{\neg A}(u) = 1 - \mu_A(u)$$

NOTES

Standard Union

$\mu_{A\cap B}(u) = \min\{\mu_A(u), \mu_B(u)\}$

In general, the triple (i, u, n) is called De Morgan Triplet if

- i is a t-norm,
- u is a t-conorm (also known as s-norm),
- n is a strong negator, so that for all $x, y \in [0, 1]$ the following holds true: $u(x,y) = n(i\,(n(x), n(y)))$ generalized De Morgan relation. This implies the axioms provided below in detail.

Aggregation Operations

Aggregation operations on fuzzy sets are operations by which several fuzzy sets are combined in a desirable way to produce a single fuzzy set.

Aggregation operation on n fuzzy set ($2 \leq n$) is defined by a function

h: [0, 1] $n \rightarrow$ [0, 1]

Axioms for Aggregation Operations Fuzzy Sets

Axiom *h*1. Boundary condition

$h(0, 0, ..., 0) = 0$ and $h(1, 1, ..., 1) = 1$

Axiom *h*2. Monotonicity

For any pair <$a1, a2, ..., an$> and <$b1, b2, ..., bn$> of n-tuples such that $ai, bi \in$ [0,1] for all $i \in$ Nn, if $ai \leq bi$ for all $i \in$ Nn, then $h(a1, a2, ...,an) \leq h(b1, b2, ..., bn)$; that is, h is monotonic increasing in all its arguments.

Axiom *h*3. Continuity

h is a continuous function.

2.3 FUZZY COMPLEMENT

Fuzzy Complement Compliment of a fuzzy set A is the set of elements of X excluding those of set A = $\{x \mid x \notin A\}$ In membership function: μ A(x) = $\{x \in X \mid$ 1- A(x)$\}$ A complement of a fuzzy set A is specified by a function c: [0,1] $\rightarrow$ [0,1], which assigns a value $c(\mu$ A(x)) to each membership grade μ A(x).

Definition: Let X be a non-zero set. A fuzzy set A of this set X is defined by the following set of pairings. A= (x, $A(x)$): μX, where A: X [0, 1] is the grade of membership or degree of belongings or degree of membership of μX in A and $A\,x$ is the grade of membership or degree of belongings or degree of membership of μX in A. As a result, a fuzzy set is a collection of pairs consisting of a given universe element and its degree of membership. A can also be represented as A = (x1, A(x1)), (x2, A(x2)),.......(x n, A(x n)).

NOTES

The membership function of a fuzzy set can take any value between 0 and 1. If the membership function of a fuzzy set is identically zero, it is said to be empty. A fuzzy set is universal if its membership function x is the same on all X, i.e., x(X) = Two fuzzy sets are equal if A(x) = B(x), μX. A subset of a fuzzy set A(X) is said to be a subset of a fuzzy set B(X), A(x) B(x), μ = X.

Support of a Fuzzy Set: The support of a fuzzy set A is S(A) which is a crisp set $\forall$ x ∈ X, such that μ A(x) > 0. S(A) = {x ∈ X | μ A(x) > 0}.

The α level set or α cut: The α level set is a crisp set of elements that belongs to the fuzzy set A at least to the degree α i.e., A a = {x ∈ X | m A(x) ≥ α} Strong α Level set: The strong α level set is defined as A a′ {x ∈ X | μ A(x) > α}, when an element of a fuzzy set achieves the highest attainable membership grade, it is called a normalised fuzzy set. Equation of a characteristic: A(x) = 1, x A2(x) = 0, x A2(x) = 0, x A2(x) = 0, x A2(x) = 0, x A2(x) = 0, x A2(x) = 0, x A2(x) = 0, x A2(x) = 0, x A2(x) = It gives a member the value 1 and a non-member the value 0. As a result, it's also known as the membership function. The cardinality of a set is indicated by A and is defined as the number of elements in the set.

An Extraordinary: The complement of a fuzzy set A is the set of X items that do not belong to set A. A = x | x A = x A = x A = x A = x A = x A = x In the function of membership: 1- A(x) = x X A(x) = x X A(x) = x X A(x) = x X A(x) = x X A(x) = x A function c: [0,1] specifies the complement of a fuzzy set A, assigning a value c(A(x)) to each membership grade A(x). The membership grade of the element x in the fuzzy set reflecting the negation of the notion represented by A is understood as this assigned value. As a result, if A is a fuzzy set of tall men, its counterpart is a fuzzy set of short men. Obviously, many elements in both a fuzzy set and its complement can have some nonzero degree of membership.

μA(x) is defined as the degree to which x belongs to A. Let CA denote a fuzzy complement of A of type c. Then μCA(x) is the degree to which x belongs to CA, and the degree to which x does not belong to A. (μA(x) is therefore the degree to which x does not belong to CA.) Let a complement CA be defined by a function

c: [0,1] → [0,1]

for all $x \in$ U: μCA(x) = c(μA(x))

Axioms for Fuzzy Complements

Axiom ***c*1. Boundary condition**

$c(0) = 1$ and $c(1) = 0$

Axiom *c*2. Monotonicity

For all $a, b \in [0, 1]$, if $a < b$, then $c(a) > c(b)$

Axiom *c*3. Continuity

c is continuous function.

Axiom c4. Involutions

c is an involution, which means that $c(c(a)) = a$ for each $a \in [0,1]$

c is a strong negator (also known as fuzzy complement).

A function c satisfying axioms c1 and c2 has at least one fixpoint a^* with

$c(a^*) = a^*$, and if axiom $c3$ is fulfilled as well there is exactly one such fixpoint. For the standard negator $c(x) = 1\text{-}x$ the unique fixpoint is $a^* = 0.5$.

2.4 FUZZY UNIONS

The union of two fuzzy sets A and B is specified in general by a binary operation on the unit interval function of the form

u:$[0,1]\times[0,1] \rightarrow [0,1]$.

For all $x \in$ U: $\mu A \cup B(x) = u[\mu A(x), \mu B(x)]$.

Axioms for Fuzzy Union

Axiom u1. Boundary condition

$u(a, 0) = u(0, a) = a$

Axiom u2. Monotonicity

$b \leq d$ implies $u(a, b) \leq u(a, d)$

Axiom u3. Commutativity

$u(a, b) = u(b, a)$

Axiom u4. Associativity

$u(a, u(b, d)) = u(u(a, b), d)$

Axiom u5. Continuity

u is a continuous function

Axiom u6. Super Idempotency

$u(a, a) \leq a$

Axiom u7. Strict monotonicity

$a1 < a2$ and $b1 < b2$ implies $u(a1, b1) < u(a2, b2)$

Axioms u1 up to u4 define a t-conorm (also known as s-norm or fuzzy intersection). The standard t-conorm max is the only idempotent t-conorm (i.e., $u(a1, a1) = a$ for all $a \in [0, 1]$).

Fuzzy Intersections

The intersection of two fuzzy sets A and B is specified in general by a binary operation on the unit interval, a function of the form

i: $[0,1] \times [0,1] \rightarrow [0,1]$.

For all $x \in$ U: $\mu A \cap B(x) = i[\mu A(x), \mu B(x)]$.

Axioms for Fuzzy Intersection

Axiom *i*1. Boundary condition

$i(a, 1) = a$

Axiom *i*2. Monotonicity

$b \le d$ implies $i(a, b) \le i(a, d)$

Axiom *i*3. Commutativity

$i(a, b) = i(b, a)$

Axiom *i*4. Associativity

$i(a, i(b, d)) = i(i(a, b), d)$

Axiom *i*5. Continuity

i is a continuous function

Axiom *i*6. Sub Idempotency

$i(a, a) \le a$

Axiom *i*7. Strict monotonicity

$i(a1, b1) \le i(a2, b2)$ if $a1 \le a2$ and $b1 \le b2$

Axioms $i1$ up to $i4$ define a t-norm (also known as fuzzy intersection). The standard t-norm min is the only idempotent t-norm (that is, $i(a1, a1) = a$ for all $a \in [0, 1]$).

Check Your Progress

1. Define standard fuzzy set operations.
2. What is meant by fuzzy complement?
3. Elaborate on the fuzzy unions.
4. What do you understand by fuzzy intersections?

2.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Let A and B be fuzzy sets that A, B $\subseteq$ U, u is any element in the U universe: $u \in$ U.

   Standard Complement

   $$\mu_{\neg A}(u) = 1 - \mu_A(u)$$

   Standard Union

   $$\mu_{A \cap B}(u) = \min\{\mu_A(u), \mu_B(u)\}$$

NOTES

2. Fuzzy complement compliment of a fuzzy set A is the set of elements of X excluding those of set A, μ A = $\{x \mid x \notin A\}$ In membership function: μ A(x) = $\{x \in X \mid 1- \mu A(x)\}$ A complement of a fuzzy set A is specified by a function c: $[0,1] \rightarrow [0,1]$, which assigns a value $c(\mu A(x))$ to each membership grade μ A(x).
3. The union of two fuzzy sets A and B is specified in general by a binary operation on the unit interval function of the form

   u:$[0,1]\times[0,1] \rightarrow [0,1]$.

   For all $x \in$ U: $\mu A \in B(x) = u[\mu A(x), \mu B(x)]$.
4. The intersection of two fuzzy sets A and B is specified in general by a binary operation on the unit interval, a function of the form

   i: $[0,1] \times [0,1] \rightarrow [0,1]$.

   For all $x \in$ U: $\mu A \cap B(x) = i[\mu A(x), \mu B(x)]$.

2.6 SUMMARY

- A fuzzy set operation is an operation on fuzzy sets. These operations are generalization of crisp set operations.
- The most widely used operations are called standard fuzzy set operations.
- Let A and B be fuzzy sets that A, B $\subseteq$ U, u is any element in the U universe: $u \in$ U.

  Standard complement

  $$\mu_{\neg A}(u) = 1 - \mu_A(u)$$
- Aggregation operations on fuzzy sets are operations by which several fuzzy sets are combined in a desirable way to produce a single fuzzy set.
- Pair <a1, a2, ..., an> and <b1, b2, ..., bn> of n-tuples such that ai, $bi \in$ [0,1] for all $i \in$ Nn, if $ai \leq bi$ for all $i \in$ Nn, then $h(a1, a2, ...,an) \leq h(b1, b2, ..., bn)$; that is, h is monotonic increasing in all its arguments.
- Fuzzy Complement Compliment of a fuzzy set A is the set of elements of X excluding those of set A, A = $\{x \mid x \notin A\}$ In membership function: m
- A(x) = $\{x \in X \mid 1- \mu A(x)\}$ A complement of a fuzzy set A is specified by a function c: $[0,1] \rightarrow [0,1]$, which assigns a value $c\mu$ A(x)) to each membership grade μ A(x).
- A fuzzy set is a collection of pairs consisting of a given universe element and its degree of membership.
- A fuzzy set is universal if its membership function x is the same on all X, i.e., x(X) = Two fuzzy sets are equal if A(x) = B(x), μX. A subset of a fuzzy set A(X) is said to be a subset of a fuzzy set B(X), A(x) B(x), μ = X.

NOTES

- When an element of a fuzzy set achieves the highest attainable membership grade, it is called a normalised fuzzy set.
- The cardinality of a set is indicated by A and is defined as the number of elements in the set.
- The union of two fuzzy sets A and B is specified in general by a binary operation on the unit interval function of the form

  u:[0,1]×[0,1] → [0,1].
- The intersection of two fuzzy sets A and B is specified in general by a binary operation on the unit interval, a function of the form

  i: [0,1] × [0,1] → [0,1].

  For all $x \notin$ U: μA ∩ B(x) = i[μA(x), μB(x)].
- Axioms i1 up to i4 define a t-norm. The standard t-norm min is the only idempotent t-norm (that is, i (a1, a1) = a for all $a \in$ [0, 1]).

2.7 KEY WORDS

- **Aggregation operation:** Aggregation operations on fuzzy sets are operations by which several fuzzy sets are combined in a desirable way to produce a single fuzzy set.
- **Fuzzy complement:** Fuzzy complement compliment of a fuzzy set A is the set of elements of X excluding those of set A, A = {x | $x \notin$ A} In membership function: μ A(x) = {$x \in$ X | 1- μ A(x)} A complement of a fuzzy set A is specified by a function c: [0,1] → [0,1], which assigns a value cμ A(x)) to each membership grade μ A(x).
- **Normalised fuzzy set:** When an element of a fuzzy set achieves the highest attainable membership grade, it is called a normalised fuzzy set.
- **Fuzzy unions:** The union of two fuzzy sets A and B is specified in general by a binary operation on the unit interval function of the form

  u:[0,1]×[0,1] → [0,1].

  For all $x \in$ U: μA ∩ B(x) = u[μA(x), μB(x)].
- **Fuzzy intersections:** The intersection of two fuzzy sets A and B is specified in general by a binary operation on the unit interval, a function of the form i: [0, 1] × [0, 1] → [0, 1].

  For all $x \in$ U: μA ∩ B(x) = i[μA(x), μB(x)].

2.8 SELF-ASSESSMENT QUESTIONS AND EXERCISES

NOTES

Short-Answer Questions

1. Define standard fuzzy set operations with examples.
2. Explain fuzzy complement.
3. What do you understand by the normalised fuzzy set?
4. Define the term fuzzy unions.
5. Describe the fuzzy intersections.

Long-Answer Questions

1. What are fuzzy set? Discuss the types of fuzzy set giving examples of each type.
2. Explain briefly about the concept of aggregation operation giving suitable examples of each.
3. What are fuzzy complement? Explain the axioms for fuzzy complements.
4. Describe briefly the concept of fuzzy unions.
5. Briefly discuss about the fuzzy intersections giving appropriate examples.

2.9 FURTHER READINGS

Klir, George J. and Bo Yuan. 1995. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. New Jersey: Prentice Hall Inc.

Klir, George J. and Tina A. Folger. 2009. *Fuzzy Sets, Uncertainty and Information*. New Delhi: Prentice Hall of India.

Zimmermann, Hans-Jürgen. 1991. *Fuzzy Set Theory—And Its Applications*. New Delhi: Allied Publishers Private Limited.

UNIT 3 COMBINATIONS AND FUZZY NUMBERS

NOTES

Structure

3.0 INTRODUCTION

Applied Fuzzy Arithmetic is a well-organized compendium that provides both a deeper understanding of fuzzy arithmetic theory and a broad view of its applications in the engineering sciences, making it a valuable resource for students, researchers, and practising engineers. The first half of the book is an introduction to fuzzy arithmetic theory, with the goal of making the material well-organized and understandable. Existing formulations of fuzzy arithmetic are provided and integrated in the general framework, as is the derivation of fuzzy arithmetic from the original fuzzy set theory and its progression towards a practical implementation.

A fuzzy number is a generalization of a regular, real number in the sense that it does not refer to one single value but rather to a connected set of possible values, where each possible value has its own weight between 0 and 1. This weight is called the membership function. A fuzzy number is thus a special case of a convex, normalized fuzzy set of the real line. Just like fuzzy logic is an extension of Boolean logic, fuzzy numbers are an extension of real numbers. Calculations with fuzzy numbers allow the incorporation of uncertainty on parameters, properties, geometry, initial conditions, etc. The arithmetic calculations on fuzzy numbers are implemented using fuzzy arithmetic operations, which can be done by two different approaches: interval arithmetic approach; and the extension principal approach. A fuzzy number is equal to a fuzzy interval. The degree of fuzziness is determined by the a-cut which is also called the fuzzy spread.

In this unit, you will study about the combinations of operations, fuzzy arithmetic and fuzzy number.

3.1 OBJECTIVES

NOTES

After going through this unit, you will be able to:

- Describe the basics concepts of combinations of operations
- Explain the fuzzy arithmetic
- Understand the basic concepts of fuzzy number

3.2 COMBINATIONS OF OPERATIONS

Applied Fuzzy Arithmetic is a well-organized compendium that provides both a deeper understanding of fuzzy arithmetic theory and a broad view of its applications in the engineering sciences, making it a valuable resource for students, researchers, and practising engineers. The first half of the book is an introduction to fuzzy arithmetic theory, with the goal of making the material well-organized and understandable. Existing formulations of fuzzy arithmetic are provided and integrated in the general framework, as is the derivation of fuzzy arithmetic from the original fuzzy set theory and its progression towards a practical implementation.

Different types of operations can be combined within the same operational expression. Any combination can be used. Consider the following example:

Each operation within the expression is evaluated according to the rules for that kind of operation, with necessary data conversions taking place before the operation is performed, as follows:

- The decimal value of A is converted to binary base.
- The binary addition is performed, adding A and B.
- The binary result is compared with the converted binary value of C.
- The bit result of the comparison is extended to the length of the bit variable D and the AND operation is performed.
- The result of the AND operation, a bit string of length 4, is assigned to Result without conversion, but with truncation on the right.

The expression in this example is evaluated operation-by-operation, from left to right. The order of evaluation, however, depends upon the priority of the operators appearing in the expression.

Operational Expressions

An operational expression consists of one or more single operations. A single operation is either a prefix operation (an operator preceding a single operand) or an infix operation (an operator between two operands). The two operands of any infix operation normally should be the same data type when the operation is performed.

NOTES

The operands of an operation in a PL/I (Programming Language) expression are converted, if necessary, to the same data type before the operation is performed.

There are few restrictions on the use of different data types in an expression.

However, these mixtures imply conversions. If conversions take place at run time, the program takes longer to run. Also, conversion can result in loss of precision. When using expressions that mix data types, you must understand the relevant conversion rules.

The classes of operations include handle, pointer, arithmetic, bit, comparison, and concatenation.

Handle Operations

These handle operations can be used in PL/I programs.

Pointer Operations

These pointer operations can be used in PL/I programs.

Arithmetic Operations

An arithmetic operation is specified by combining operands with one arithmetic operator. Arithmetic operations can also be specified by the ADD, SUBTRACT, DIVIDE, and MULTIPLY built-in functions.

Bit Operations

A bit operation is specified by combining operands with a logical operator.

Combinations of Operations

Different types of operations can be combined within the same operational expression. Any combination can be used.

Each operation in the expression is assessed according to the rules for that operation, with any necessary data conversions occurring before the operation is carried out, as follows:

- The binary value of A is transformed from its decimal value.
- The binary addition is done by adding A and B together.
- The converted binary value of C is compared to the binary result.
- The AND operation is done on the bit result of the comparison, which is extended to the length of the bit variable D.
- Without conversion, but with right-hand truncation, the result of the AND operation, a bit string of length 4, is assigned to Result.

Expressions and References

An expression is a representation of a value. An expression can be one of the following:

- A single constant, variable or function reference.
- Any combination of constants, variables, or function references, including operators and parentheses used in the combination.
- An expression that contains operators is an operational expression.
- The constants and variables of an operational expression are called operands.

An Element Expression

Represents a single value. This definition includes an elementary name within a structure or a union or a subscripted name that specifies a single element of an array.

An Array Expression

Represents an array of values. This definition includes a member of a structure or union that has the dimension attribute.

A Structure Expression

Represents a structured set of values.

The syntax of many PL/I statement allows expressions, provided the result of the expression conforms to the syntax rules. Unless specifically stated in the text following the syntax specification, the unqualified term expression or reference refers to a scalar expression. For expressions other than a scalar expression, the type of expression is noted. For example, the term array expression indicates that a scalar expression is not valid.

Here is an example of a structure expression:

Rate = Rate × 2q

- **Order of Evaluation**

PL/I statement often contain more than one expression or reference. Except as described for specific instances (for example, the assignment statement), evaluation can be in any order, or (conceptually) at the same time.

- **Targets**

The results of an expression evaluation or of a conversion are assigned to a target. Targets can be variables, pseudo variables, or intermediate results.

- **Operational Expressions**

An operational expression consists of one or more single operations. A single operation is either a prefix operation (an operator preceding a single operand) or an infix operation (an operator between two operands). The two operands of any infix operation normally should be the same data type when the operation is performed.

NOTES

- **Array Expressions**

Array expressions can include operators (both prefix and infix), element variables, and constants. The rules for combining operations and for data conversion of operands are the same as for element operations.

- **Structure Expressions**

Structure expressions, unlike structure references, are allowed only in assignments and as arguments to procedures or functions, as long as the associated parameter has constant extents, namely, constant string lengths, area sizes and array bounds.

- **Restricted Expressions**

Where PL/I require a (possibly signed) constant, a restricted expression can be used.

3.3 FUZZY ARITHMETIC

The arithmetic calculations on fuzzy numbers are implemented using fuzzy arithmetic operations, which can be done by two different approaches:

(1) Interval Arithmetic Approach

(2) The Extension Principal Approach

The degree of fuzziness is determined by the α-cut which is also called the fuzzy spread.

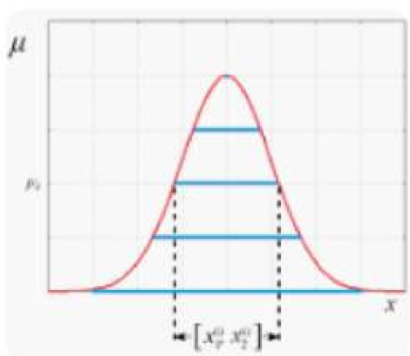

Fig. 3.1 *Fuzzy Spread*

Fuzzy Interval Arithmetic

Interval arithmetic can also be used with affiliation functions for fuzzy quantities as they are used in fuzzy logic. Apart from the strict statements

$x \in [x]$ and $x \notin [x]$,

Intermediate values are also possible, to which real numbers $\mu \in [0,1]$ are assigned. $\mu = 1$ corresponds to definite membership while $\mu = 0$ is non-membership. A distribution function assigns uncertainty, which can be understood as a further interval.

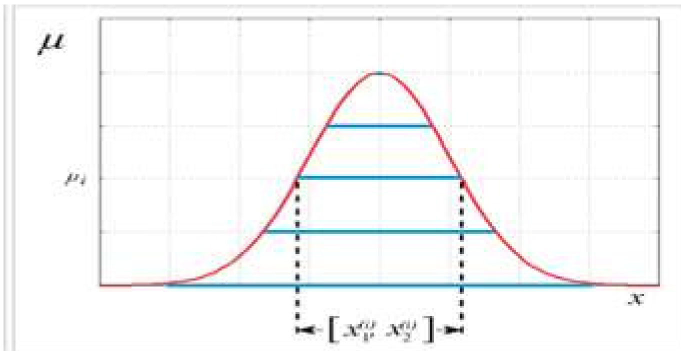

Fig. 3.2 *Approximation of the Normal Distribution by a Sequence of Intervals*

For fuzzy arithmetic only a finite number of discrete affiliation stages $\mu_i \in [0,1]$ are considered. The form of such a distribution for an indistinct value can then represented by a sequence of intervals

$$\left[x^{(1)}\right] \supset \left[x^{(2)}\right] \supset \cdots \supset \left[x^{(k)}\right].$$

The interval $[x^i]$ corresponds exactly to the fluctuation range for the stage μ_i. $f(x_1, \ldots, x_n)$The appropriate distribution for a function concerning indistinct values $x_1, \ldots x_n$ and the corresponding sequences

$$\left[x_1^{(1)}\right] \supset \cdots \supset \left[x_1^{(k)}\right], \ldots, \left[x_n^{(1)}\right] \supset \cdots \supset \left[x_n^{(k)}\right]$$

can be approximated by the sequence

$$\left[y^{(1)}\right] \supset \cdots \supset \left[y^{(k)}\right],$$

Where,

$$\left[y^{(i)}\right] = f\left(\left[x_1^{(i)}\right], \ldots \left[x_n^{(i)}\right]\right)$$

And can be calculated by interval methods. The value $[y^{(1)}]$ corresponds to the result of an interval calculation.

3.4 FUZZY NUMBER

A fuzzy number is a generalization of a regular, real number in the sense that it does not refer to one single value but rather to a connected set of possible values, where each possible value has its own weight between 0 and 1. This weight is called the membership function. A fuzzy number is thus a special case of a convex, normalized fuzzy set of the real line. Just like fuzzy logic is an extension of Boolean logic (which uses absolute truth and falsehood only and nothing in between), fuzzy numbers are an extension of real numbers. Calculations with fuzzy numbers allow the incorporation of uncertainty on parameters, properties, geometry, initial conditions, etc. The arithmetic calculations on fuzzy numbers are implemented using fuzzy arithmetic operations, which can be done by two different approaches: interval arithmetic approach; and the extension principal approach.

A fuzzy number is equal to a fuzzy interval. The degree of fuzziness is determined by the α-cut which is also called the fuzzy spread.

Membership Function

The membership function of a fuzzy set is a generalization of the indicator function for classical sets. In fuzzy logic, it represents the degree of truth as an extension of valuation. Degrees of truth are often confused with probabilities, although they are conceptually distinct, because fuzzy truth represents membership in vaguely defined sets, not likelihood of some event or condition. Membership functions were introduced by Zadeh in the first paper on fuzzy sets (1965). Zadeh, in his theory of fuzzy sets, proposed using a membership function (with a range covering the interval (0, 1)) operating on the domain of all possible values.

Definition

For any set X a membership function on X is any function from to the real unit interval [0, 1].

Membership functions represent fuzzy subsets of X. The membership function which represents a fuzzy set $\tilde{A}$ is usually denoted by μ_A. For an element x of X the value $\mu_A(x)$ is called the membership degree of x in the fuzzy set $\tilde{A}$. The membership degree $\mu_A(x)$ quantifies the grade of membership of the element x to the fuzzy set $\tilde{A}$. The value 0 means that x is not a member of the fuzzy set; the value 1 means that x is fully a member of the fuzzy set. The values between 0 and 1 characterize fuzzy members, which belong to the fuzzy set only partially.

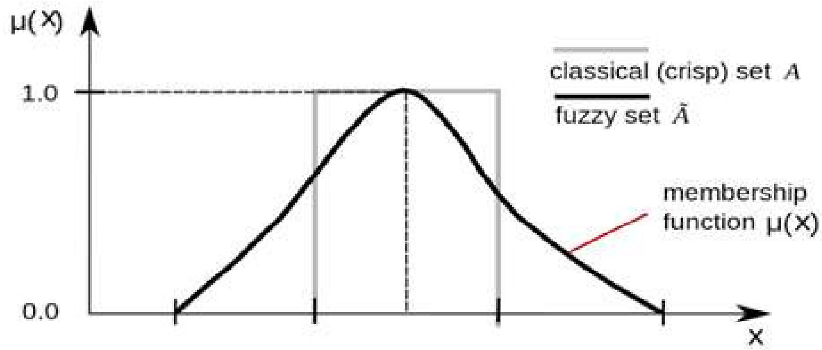

Fig. 3.3 *Membership of a Fuzzy Set*

Sometimes, a more general definition is used, where membership functions take values in an arbitrary fixed algebra or structure L usually it is required that *L* be at least a poset or lattice. The usual membership functions with values in [0, 1] are then called [0, 1]-valued membership functions.

Capacity

One application of membership functions is as capacities in decision theory $\nu(\emptyset) = 0, \nu(\Omega) = 1.$

In decision theory, a capacity is defined as a function, *v* from *S*, one application of membership functions is as capacities in decision theory. In decision theory, a capacity is defined as a function, [0, 1] such that *v* is set-wise monotone and is normalized.

This is a generalization of the notion of a probability measure, where the probability axiom of countable additively is weakened. A capacity is used as a subjective measure of the likelihood of an event, and the 'Expected Value' of an outcome given a certain capacity can be found by taking the Choquet integral over the capacity.

Fuzzy Measure Theory

Fuzzy measure theory considers generalized measures in which the additive property is replaced by the weaker property of monotonicity. The central concept of fuzzy measure theory is the fuzzy measure which was introduced by Choquet in 1953 and independently defined by Sugeno in 1974 in the context of fuzzy integrals. There exists a number of different classes of fuzzy measures including plausibility/belief measures; possibility/necessity measures; and probability measures which are a subset of classical measures.

Definitions

Let X be a universe of discourse, be a class of subsets of *C* can be subset of X and $E, F \in \mathcal{C}$.

$g : \mathcal{C} \to \mathbb{R}$ a function

Where,

NOTES

1. $\emptyset \in \mathcal{C} \Rightarrow g(\emptyset) = 0$
2. $E \subseteq F \Rightarrow g(E) \leq g(F)$

$g(\mathbf{X}) = 1.$ is called a fuzzy measure. A fuzzy measure is called normalized or regular if g(X) = 1

Check Your Progress

1. What are operational expressions?
2. Define fuzzy arithmetic.
3. What do you understand by fuzzy numbers?
4. Explain the basic concept of membership function.

3.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

- An operational expression consists of one or more single operations. A single operation is either a prefix operation (an operator preceding a single operand) or an infix operation (an operator between two operands). The two operands of any infix operation normally should be the same data type when the operation is performed.
- The arithmetic calculations on fuzzy numbers are implemented using fuzzy arithmetic operations, which can be done by two different approaches:

  (1) Interval arithmetic approach

  (2) The extension principal approach

  The degree of fuzziness is determined by the a-cut which is also called the fuzzy spread.
- A fuzzy number is a generalization of a regular, real number in the sense that it does not refer to one single value but rather to a connected set of possible values, where each possible value has its own weight between 0 and 1.
- Membership functions were introduced by Zadeh in the first paper on fuzzy sets (1965). Zadeh, in his theory of fuzzy sets, proposed using a membership function (with a range covering the interval (0,1)) operating on the domain of all possible values.

3.6 SUMMARY

NOTES

- Each operation within the expression is evaluated according to the rules for that kind of operation, with necessary data conversions taking place before the operation is performed.
- An operational expression consists of one or more single operations. A single operation is either a prefix operation (an operator preceding a single operand) or an infix operation (an operator between two operands). The two operands of any infix operation normally should be the same data type when the operation is performed.
- The operands of an operation in a PL/I expression are converted, if necessary, to the same data type before the operation is performed.
- An arithmetic operation is specified by combining operands with one arithmetic operator. Arithmetic operations can also be specified by the ADD, SUBTRACT, DIVIDE, and MULTIPLY built-in functions.
- An elementary name within a structure or a union or a subscripted name that specifies a single element of an array.
- A member of a structure or union that has the dimension attribute.
- An operational expression consists of one or more single operations. A single operation is either a prefix operation (an operator preceding a single operand) or an infix operation (an operator between two operands). The two operands of any infix operation normally should be the same data type when the operation is performed.
- Array expressions can include operators (both prefix and infix), element variables, and constants. The rules for combining operations and for data conversion of operands are the same as for element operations.
- Structure expressions, unlike structure references, are allowed only in assignments and as arguments to procedures or functions, as long as the associated parameter has constant extents, namely, constant string lengths, area sizes and array bounds.
- The arithmetic calculations on fuzzy numbers are implemented using fuzzy arithmetic operations, which can be done by two different approaches are interval arithmetic approach and the extension principal approach. The degree of fuzziness is determined by the α-cut which is also called the fuzzy spread.
- A fuzzy number is a generalization of a regular, real number in the sense that it does not refer to one single value but rather to a connected set of possible values, where each possible value has its own weight between 0 and 1.

NOTES

- The membership function of a fuzzy set is a generalization of the indicator function for classical sets. In fuzzy logic, it represents the degree of truth as an extension of valuation.
- Membership functions were introduced by Zadeh in the first paper on fuzzy sets (1965). Zadeh, in his theory of fuzzy sets, proposed using a membership function (with a range covering the interval (0,1)) operating on the domain of all possible values.

3.7 KEY WORDS

- **Operational expressions:** An operational expression consists of one or more single operations. A single operation is either a prefix operation (an operator preceding a single operand) or an infix operation (an operator between two operands). The two operands of any infix operation normally should be the same data type when the operation is performed.
- **Array expressions:** Array expressions can include operators (both prefix and infix), element variables and constants. The rules for combining operations and for data conversion of operands are the same as for element operations.
- **Structure expressions:** Structure expressions, unlike structure references, are allowed only in assignments and as arguments to procedures or functions, as long as the associated parameter has constant extents, namely, constant string lengths, area sizes and array bounds.
- **Fuzzy arithmetic:** The arithmetic calculations on fuzzy numbers are implemented using fuzzy arithmetic operations, which can be done by two different approaches are interval arithmetic approach and the extension principal approach. The degree of fuzziness is determined by the α-cut which is also called the fuzzy spread.
- **Fuzzy number:** A fuzzy number is a generalization of a regular, real number in the sense that it does not refer to one single value but rather to a connected set of possible values, where each possible value has its own weight between 0 and 1.
- **Membership function:** Membership functions were introduced by Zadeh in the first paper on fuzzy sets (1965). Zadeh, in his theory of fuzzy sets, proposed using a membership function (with a range covering the interval (0, 1)) operating on the domain of all possible values.

3.8 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. What do you understand by the combinations of operations?
2. Describe the term fuzzy arithmetic.
3. Give the definition of fuzzy number.

Long-Answer Questions

1. What are combinations of operations? Explain giving examples.
2. Discuss briefly about the fuzzy arithmetic giving appropriate examples.
3. Explain briefly about the fuzzy number with the help of suitable examples.

3.9 FURTHER READINGS

Klir, George J. and Bo Yuan. 1995. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. New Jersey: Prentice Hall Inc.

Klir, George J. and Tina A. Folger. 2009. *Fuzzy Sets, Uncertainty and Information*. New Delhi: Prentice Hall of India.

Zimmermann, Hans-Jürgen. 1991. *Fuzzy Set Theory—And Its Applications*. New Delhi: Allied Publishers Private Limited.

UNIT 4 ARITHMETIC OPERATIONS

NOTES

Structure

4.0 INTRODUCTION

Interval arithmetic (also known as interval mathematics, interval analysis or interval computation) is a mathematical technique used to put bounds on rounding errors and measurement errors in mathematical computation. Numerical methods using interval arithmetic can guarantee reliable, mathematically correct results. Instead of representing a value as a single number, interval arithmetic represents each value as a range of possibilities. For example, instead of estimating the height of someone as exactly 2.0 metres, using interval arithmetic one might be certain that person is somewhere between 1.97 and 2.03 metres.

Mathematically, instead of working with an uncertain real x, one works with the ends of an interval $[a, b]$ that contains x. In interval arithmetic, any variable x lies in the closed interval between a and b. A function f, when applied to x, yields an uncertain result; f produces an interval $[c, d]$ which includes all the possible values for $f(x)$ for all $x \in [a, b]$.

The arithmetic operators on fuzzy numbers are basic content in fuzzy mathematics. Multiplication operation on fuzzy numbers is defined by the extension principle. The procedure of addition or subtraction is simple, but the procedure of multiplication or division is complex.

In this unit, you will study about the arithmetic operations on intervals and arithmetic operations on fuzzy numbers.

4.1 OBJECTIVES

After going through this unit, you will be able to:

- Describe the basic concept of arithmetic operations on intervals
- Discuss about the arithmetic operations on fuzzy numbers

4.2 ARITHMETIC OPERATIONS ON INTERVALS

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Interval Arithmetic

Interval arithmetic (also known as interval mathematics, interval analysis or interval computation) is a mathematical technique used to put bounds on rounding errors and measurement errors in mathematical computation. Numerical methods using interval arithmetic can guarantee reliable, mathematically correct results. Instead of representing a value as a single number, interval arithmetic represents each value as a range of possibilities. For example, instead of estimating the height of someone as exactly 2.0 metres, using interval arithmetic one might be certain that that person is somewhere between 1.97 and 2.03 metres.

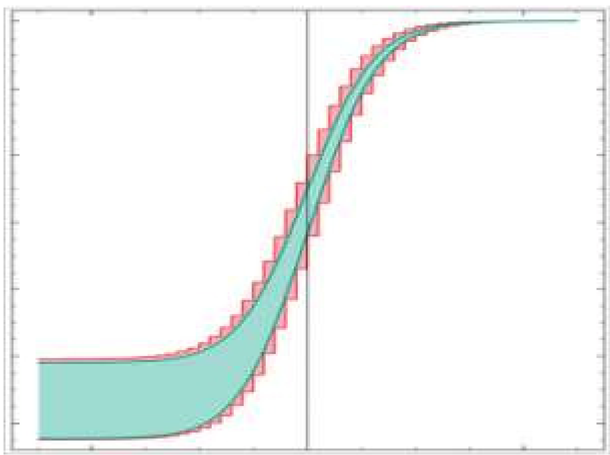

Fig. 4.1 *Tolerance Function and Interval-Valued Approximation*

Mathematically, instead of working with an uncertain real x, one works with the ends of an interval [a. b] that contains x. In interval arithmetic, any variable x lies in the closed interval between a and b A function f, when applied to x, yields an uncertain result; f produces an interval [c, d] which includes all the possible values for $f(x)$ for all $x \in [a, b]$.

Interval arithmetic is suitable for a variety of purposes. The most common use is in software, to keep track of rounding errors in calculations and of uncertainties in the knowledge of the exact values of physical and technical parameters. The latter often arise from measurement errors and tolerances for components or due to limits on computational accuracy. Interval arithmetic also helps find guaranteed solutions to equations (differential equations) and optimization problems.

Introduction

The main objective of interval arithmetic is a simple way to calculate upper and lower bounds for the range of a function in one or more variables. These endpoints

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are not necessarily the true supremum or infimum, since the precise calculation of those values can be difficult or impossible; the bounds need only contain the function's range as a subset. This treatment is typically limited to real intervals, so quantities of form

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\},$$

Where $a = -\infty$ and $b = \infty$ are allowed. With one of a, b infinite, the interval would be an unbounded interval; with both infinites, the interval would be the extended real number line. Since a real number r can be interpreted as the interval $[r, r]$ intervals and real numbers can be freely combined. As with traditional calculations with real numbers, simple arithmetic operations and functions on elementary intervals must first be defined. More complicated functions can be calculated from these basic elements.

As an example, consider the calculation of Body Mass Index (BMI) and assessing whether a person is overweight. BMI is calculated as a person's body weight in kilograms divided by the square of their height in metres. A bathroom scale may have a resolution of one kilogram. Intermediate values cannot be discerned—79.6 kg and 80.3 kg are indistinguishable, for example—but the true weight is rounded to the nearest whole number. It is unlikely that when the scale reads 80 kg, the person weighs exactly 80.0 kg. In normal rounding to the nearest value, the scale's showing 80 kg indicates a weight between 79.5 kg and 80.5 kg. This corresponds with the interval [79.5, 80.5].

For a man who weighs 80 kg and is 1.80 m tall, the BMI is approximately 24.7. A weight of 79.5 kg and the same height yields approx. 24.537, while a weight of 80.5 kg yields approx. 24.846. Since the function is monotonically increasing, we conclude that the true BMI is in the range [24.567, 24.846]. Since the entire range is less than 25, which is the cut-off between normal and excessive weight, we conclude that the man is of normal weight.

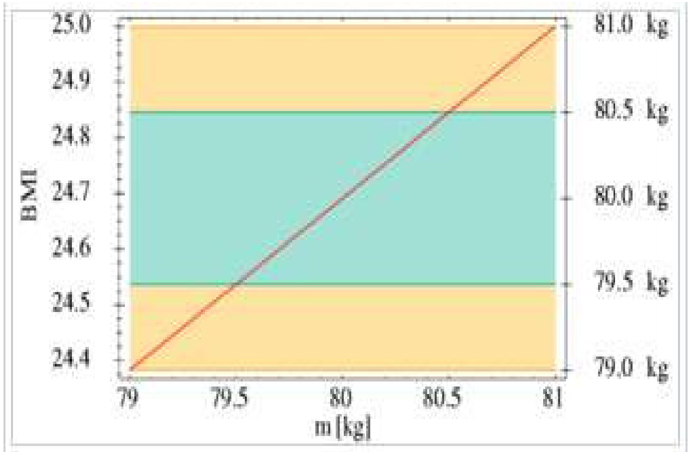

***Fig. 4.2** Body Mass Index*

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The errors in this case does not affect the conclusion (normal weight), but this is not always the case. If the man was slightly heavier, the BMI's range may include the cut-off value of 25. In that case, the scale's precision was insufficient to make a definitive conclusion. Also, note that the range of BMI examples could be reported as [24.5, 24.9] since this interval is a superset of the calculated interval. The range could not, however, be reported as [24.6, 24.8] as now the interval does not contain possible BMI values. Interval arithmetic states the range of possible outcomes explicitly. Results are no longer stated as numbers, but as intervals that represent imprecise values. The size of the intervals are similar to error bars in expressing the extent of uncertainty.

Multiple Intervals

Height and body weight both affect the value of the BMI. We have already treated weight as an uncertain measurement, but height is also subject to uncertainty. Height measurements in metres are usually rounded to the nearest centimetre: a recorded measurement of 1.79 metres actually means a height in the interval [1.785, 1.795]. Now, all four combinations of possible height/weight values must be considered. Using the interval methods described below, the BMI lies in the interval

$$\frac{[79.5, 80.5]}{[1.785, 1.795]^2} \subseteq [24.673, 25.266].$$

In this case, the man may have a normal weight or be overweight; the weight and height measurements were insufficiently precise to make a definitive conclusion. This demonstrates interval arithmetic's ability to correctly track and propagate error.

Interval Operators

A binary operation × on two intervals, such as addition or multiplication, is defined by

$$[x_1, x_2] \star [y_1, y_2] = \{x \star y \,|\, x \in [x_1, x_2] \,\wedge\, y \in [y_1, y_2]\}.$$

In other words, it is the set of all possible values of $x \times y$ where x and y are in their corresponding intervals. If $x \times y$ is monotone in each operand on the intervals, which is the case for the four basic arithmetic operations (except division when the denominator contains 0), the extreme values occur at the endpoints of the operand intervals. Writing out all combinations, one way of stating this is

$$[x_1, x_2] \star [y_1, y_2] = [\min\{x_1 \star y_1, x_1 \star y_2, x_2 \star y_1, x_2 \star y_2\}, \max\{x_1 \star y_1, x_1 \star y_2, x_2 \star y_1, x_2 \star y_2\}],$$

Provided that $x \times y$ is defined for all

For practical applications this can be simplified further:

- Addition:

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- Subtraction:
- Multiplication:

$[x_1, x_2] \cdot [y_1, y_2] = [\min\{x_1y_1, x_1y_2, x_2y_1, x_2y_2\}, \max\{x_1y_1, x_1y_2, x_2y_1, x_2y_2\}]$

- Division:

$$\frac{[x_1, x_2]}{[y_1, y_2]} = [x_1, x_2] \cdot \frac{1}{[y_1, y_2]},$$

Where,

$$\begin{aligned}
\frac{1}{[y_1, y_2]} &= \left[\tfrac{1}{y_2}, \tfrac{1}{y_1}\right] && \text{if } 0 \notin [y_1, y_2] \\
\frac{1}{[y_1, 0]} &= \left[-\infty, \tfrac{1}{y_1}\right] \\
\frac{1}{[0, y_2]} &= \left[\tfrac{1}{y_2}, \infty\right] \\
\frac{1}{[y_1, y_2]} &= \left[-\infty, \tfrac{1}{y_1}\right] \cup \left[\tfrac{1}{y_2}, \infty\right] \subseteq [-\infty, \infty] && \text{if } 0 \in (y_1, y_2)
\end{aligned}$$

The last case loses useful information about the exclusion of $(1/y_1, 1/y_2)$. Thus, it is common to work with

$\left[-\infty, \tfrac{1}{y_1}\right]$ and $\left[\tfrac{1}{y_2}, \infty\right]$

as separate intervals. More generally, when working with discontinuous functions, it is sometimes useful to do the calculation with so-called multi-intervals of the form

$\bigcup_i \, [a_i, b_i]$.

The corresponding multi-interval arithmetic maintains a set of (usually disjoint) intervals and also provides for overlapping intervals to unite.

Interval multiplication often only requires two multiplications. If x_1, y_1 are nonnegative,

$$[x_1, x_2] \cdot [y_1, y_2] = [x_1 \cdot y_1, x_2 \cdot y_2], \qquad \text{if } x_1, y_1 \geq 0.$$

The multiplication can be interpreted as the area of a rectangle with varying edges. The result interval covers all possible areas, from smallest to the largest.

It is already possible to calculate the range of simple functions, such as

$$f(a, b, x) = a \cdot x + b.$$

For example, if

$$f(a,b,x) = ([1,2]\cdot[2,3]) + [5,7] = [1\cdot 2, 2\cdot 3] + [5,7] = [7,13].$$

Notation

To make the notation of intervals smaller in formulae, brackets can be used.

$$[x] \equiv [x_1, x_2]$$

Can be used to represent an interval. Note that in such a compact notation, $[x]$ should not be confused between a single-point interval $[x_1, x_2]$ and a general interval. For the set of all intervals, we can use

$$[\mathbb{R}] := \{\, [x_1, x_2] \mid x_1 \le x_2 \text{ and } x_1, x_2 \in \mathbb{R} \cup \{-\infty, \infty\}\}$$

as an abbreviation. For a vector of intervals $([x]_1, \ldots, [x]_n) \in [\mathbb{R}]^n$

we can use a bold font: $[\mathbf{x}]$.

Elementary Functions

Interval functions beyond the four basic operators may also be defined.

For monotonic functions in one variable, the range of values is simple to compute. If $f : \mathbb{R} \to \mathbb{R}$

is monotonically increasing (respectively decreasing) in the interval $[x_1, x_2]$ then for all $y_1, y_2 \in [x_1, x_2]$

such that, y_1 d" y_2, $f(y1) < f(y_2)$ $f(y_2) < f(y_1)$

The range corresponding to the interval $[y_1, y_2] \subseteq [x_1, x_2]$

can be therefore calculated by applying the function to its endpoints:

$$f([y_1, y_2]) = [\min\{f(y_1), f(y_2)\}, \max\{f(y_1), f(y_2)\}].$$

From this, the following basic features for interval functions can easily be defined:

- Exponential function: $a^{[x_1, x_2]} = [a^{x_1}, a^{x_2}]$ for $a > 1$,
- Logarithm: $\log_a [x_1, x_2] = [\log_a x_1, \log_a x_2]$

  for positive intervals $[x_1, x_2]$ and $a > 1$,
- Even powers: For even powers, the range of values being considered is important, and needs to be dealt with before doing any multiplication.

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For example, x^n for $x \in [-1, 1]$

Should produce the interval [0, 1] when n = 2, 4, 6,… But $[-1, 1]^n$ is taken by repeating interval multiplication of form

$$[-1, 1] \cdot [-1, 1] \cdot \cdots \cdot [-1, 1]$$

Then the result is [-1, 1], wider than necessary. For piecewise monotonic functions, it is sufficient to consider the endpoints x_1, x_2 of an interval, together with the so-called critical points within the interval, being those points where the monotonicity of the function changes direction. For the sine and cosine functions, the critical points are at $\left(\frac{1}{2} + n\right)\pi$ or $n\pi$ for $n \in \mathbb{Z}$.

Thus, only up to five points within an interval need to be considered, as the resulting interval is [-1, 1] if the interval includes at least two extrema. For sine and cosine, only the endpoints need full evaluation, as the critical points lead to easily pre-calculated values—namely -1, 0, and 1.

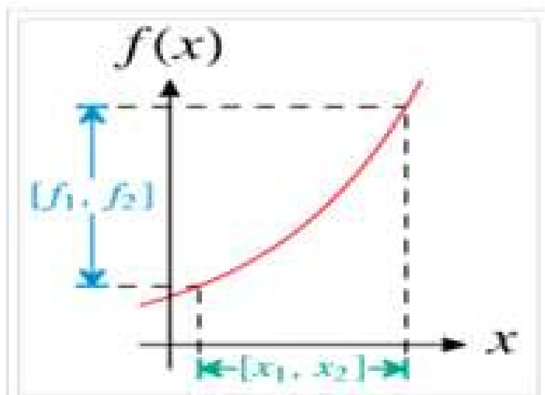

***Fig. 4.3** Values of a Monotonic Function*

Interval Extensions of General Functions

In general, it may not be easy to find such a simple description of the output interval for many functions. But it may still be possible to extend functions to interval arithmetic. If $f : \mathbb{R}^n \to \mathbb{R}$

is a function from a real vector to a real number, then

$$[f] : [\mathbb{R}]^n \to [\mathbb{R}]$$

is called an interval extension of f if $[f]([\mathbf{x}]) \supseteq \{f(\mathbf{y}) | \mathbf{y} \in [\mathbf{x}]\}$.

This definition of the interval extension does not give a precise result. For example, both

$$[f]([x_1, x_2]) = [e^{x_1}, e^{x_2}] \text{ and } [g]([x_1, x_2]) = [-\infty, \infty]$$

are allowable extensions of the exponential function. Tighter extensions are desirable, though the relative costs of calculation and imprecision should be considered; in this case, [*f*] should be chosen as it gives the tightest possible result.

Given a real expression, its natural interval extension is achieved by using the interval extensions of each of its sub expressions, functions and operators.

The Taylor interval extension (of degree *k*) is a $k + 1$ times differentiable function *f* defined by,

$$[f]([\mathbf{x}]) := f(\mathbf{y}) + \sum_{i=1}^{k} \frac{1}{i!} \mathrm{D}^i f(\mathbf{y}) \cdot ([\mathbf{x}] - \mathbf{y})^i + [r]([\mathbf{x}], [\mathbf{x}], \mathbf{y}),$$

For some $\mathbf{y} \in [\mathbf{x}]$.

Where, $\mathbf{D}^i f(\mathbf{y})$

f at the point *y* and [*r*] is an interval extension of the Taylor remainder

$$r(\mathbf{x}, \xi, \mathbf{y}) = \frac{1}{(k+1)!} \mathrm{D}^{k+1} f(\xi) \cdot (\mathbf{x} - \mathbf{y})^{k+1}.$$

The vector ξ lies between *x* and *y* with $\mathbf{x}, \mathbf{y} \in [\mathbf{x}]$, ξ

is protected by [*x*]. Usually, one chooses *y* o be the midpoint of the interval and uses the natural interval extension to assess the remainder.

The special case of the Taylor interval extension of degree $k = 0$ is also referred to as the mean value form.

Complex Interval Arithmetic

An interval can also be defined as a locus of points at a given distance from the centre, clarification needed and this definition can be extended from real numbers to complex numbers. As it is the case with computing with real numbers, computing with complex numbers involves uncertain data. So, given the fact that an interval number is a real closed interval and a complex number is an ordered pair of real numbers, there is no reason to limit the application of interval arithmetic to the measure of uncertainties in computations with real numbers. Interval arithmetic can thus be extended, via complex interval numbers, to determine regions of uncertainty in computing with complex numbers. The basic algebraic operations for real interval numbers (real closed intervals) can be extended to complex numbers. It is therefore not surprising that complex interval arithmetic is similar to, but not the same as, ordinary complex arithmetic. It can be shown that, as it is the case with real interval arithmetic, there is no distributivity between addition and

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multiplication of complex interval numbers except for certain special cases, and inverse elements do not always exist for complex interval numbers. Two other useful properties of ordinary complex arithmetic fail to hold in complex interval arithmetic: The additive and multiplicative properties, of ordinary complex conjugates, do not hold for complex interval conjugates.

Interval arithmetic can be extended, in an analogous manner, to other multidimensional number systems, such as quaternions and octonions, but with the expense that we have to sacrifice other useful properties of ordinary arithmetic.

Interval Methods

The methods of classical numerical analysis cannot be transferred one-to-one into interval-valued algorithms, as dependencies between numerical values are usually not taken into account.

Rounded Interval Arithmetic

To work effectively in a real-life implementation, intervals must be compatible with floating point computing. The earlier operations were based on exact arithmetic, but in general fast numerical solution methods may not be available. The range of values of the function

$$f(x, y) = x + y \text{ for } x \in [0.1, 0.8] \text{ and } y \in [0.06, 0.08]$$

are for example [0.16, 0.88] where the same calculation is done with single digit precision, the result would normally be [0.2, 0.9]. But [0.16,0.88].

So, this approach would contradict the basic principles of interval arithmetic, as a part of the domain of [0.1,0.8], [0.06,0.08] would be lost. Instead, the outward rounded solution [0.1, 0.9] [0.1, 0.9] is used.

The standard IEEE 754 (the IEEE standard for floating-point arithmetic) for binary floating-point arithmetic also sets out procedures for the implementation of rounding. An IEEE 754 compliant system allows programmers to round to the nearest floating point number; alternatives are rounding towards 0 (truncating), rounding toward positive infinity (i.e., up), or rounding towards negative infinity (i.e., down).

The required external rounding for interval arithmetic can thus be achieved by changing the rounding settings of the processor in the calculation of the upper limit (up) and lower limit (down). Alternatively, an appropriate small interval

$[\xi_1, \xi_2]$ can be added.

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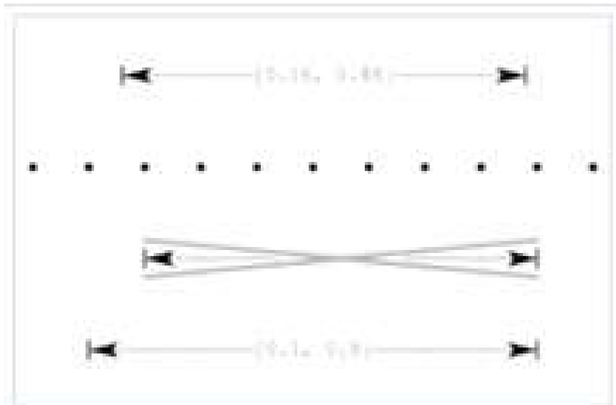

Fig. 4.4 *Outer Bounds at Different Level of Rounding*

Dependency Problem

The so-called dependency problem is a major obstacle to the application of interval arithmetic. Although interval methods can determine the range of elementary arithmetic operations and functions very accurately, this is not always true with more complicated functions. If an interval occurs several times in a calculation using parameters, and each occurrence is taken independently then this can lead to an unwanted expansion of the resulting intervals.

As an illustration, take the function f defined by $f(x) = x^2 + x$. The values of this function over the interval [-1, 1] are [-1/4, 2]. As the natural interval extension, it is calculated as:

$$[-1, 1]^2 + [-1, 1] = [0, 1] + [-1, 1] = [-1, 2],$$

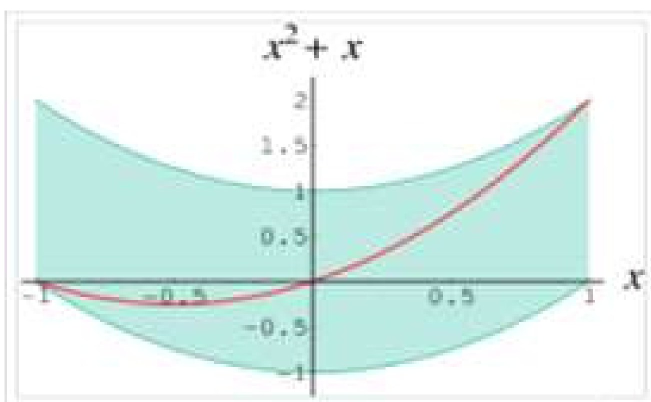

Fig. 4.5 *Approximate Estimate of the Value Range*

Which is slightly larger; we have instead calculated the infimum and supremum of the function $h(x, y) = x^2 + y$ over $x, y \in [-1, 1]$.

There is a better expression of f in which the variable x only appears once, namely by rewriting $f(x) = x^2 + x$ as addition and squaring in the quadratic.

$$f(x) = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4}.$$

NOTES

So, the suitable interval calculation is

$$\left([-1,1] + \frac{1}{2}\right)^2 - \frac{1}{4} = \left[-\frac{1}{2}, \frac{3}{2}\right]^2 - \frac{1}{4} = \left[0, \frac{9}{4}\right] - \frac{1}{4} = \left[-\frac{1}{4}, 2\right]$$

And gives the correct values.

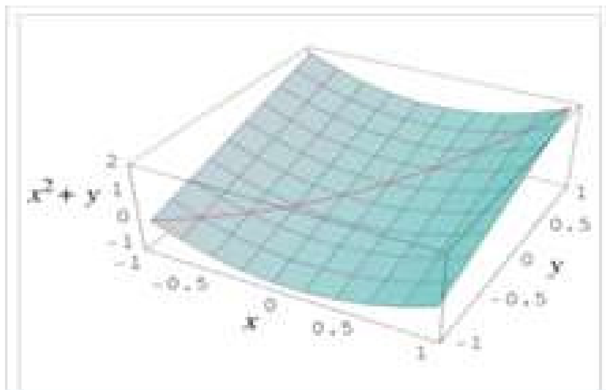

Fig. 4.6 *Treating Each Occurrence of a Variable Independently*

In general, it can be shown that the exact range of values can be achieved, if each variable appears only once and if *f* is continuous inside the box. However, not every function can be rewritten this way.

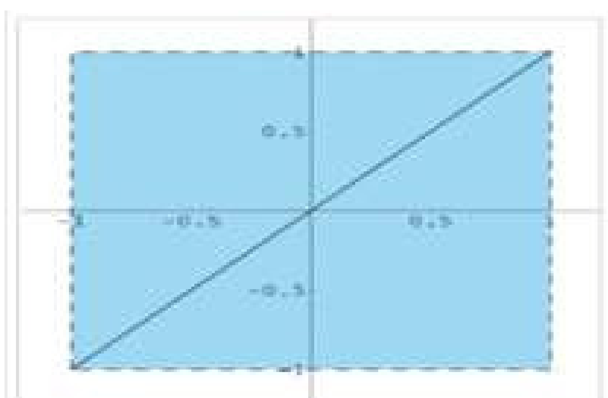

Fig. 4.7 *Wrapping Effect*

The dependency of the problem causing over-estimation of the value range can go as far as covering a large range, preventing more meaningful conclusions.

An additional increase in the range stems from the solution of areas that do not take the form of an interval vector. The solution set of the linear system

$$\begin{cases} x = p \\ y = p \end{cases} \qquad p \in [-1, 1]$$

is precisely the line between the points (-1, -1) and (1,1). Using interval methods results in the unit square, [-1×1], [-1×1] this is known as the wrapping effect.

Application

Interval arithmetic can be used in various areas (such as set inversion, motion planning, set estimation or stability analysis) to treat estimates with no exact numerical value.

Rounding Error Analysis

Interval arithmetic is used with error analysis, to control rounding errors arising from each calculation. The advantage of interval arithmetic is that after each operation there is an interval that reliably includes the true result. The distance between the interval boundaries gives the current calculation of rounding errors directly:

Error = abs (a-b) for a given interval [a, b].

Interval analysis adds to rather than substituting for traditional methods for error reduction, such as pivoting.

Tolerance Analysis

Parameters for which no exact figures can be allocated often arise during the simulation of technical and physical processes. The production process of technical components allows certain tolerances, so some parameters fluctuate within intervals. In addition, many fundamental constants are not known precisely.

If the behaviour of such a system affected by tolerances satisfies, for example,

$f(x, p) = 0$, for $\mathbf{p} \in [\mathbf{p}]$

and unknown x then the set of possible solution

$$\{\mathbf{x} \mid \exists \mathbf{p} \in [\mathbf{p}], f(\mathbf{x}, \mathbf{p}) = 0\},$$

can be found by interval methods. This provides an alternative to traditional propagation of error analysis. Unlike point methods, such as Monte Carlo simulation, interval arithmetic methodology ensures that no part of the solution area can be overlooked. However, the result is always a worst-case analysis for the distribution of error, as other probability-based distributions are not considered.

4.3 ARITHMETIC OPERATIONS ON FUZZY NUMBERS

NOTES

We'd like to refer to one of the first representations of a fuzzy set defined on a universe X (the real axis IR, say) of discourse, i.e., on the set of all possible numerical values (observations, say) of a fuzzy idea as part of the current development (say: variable or physical measurement). A fuzzy set (read: a fuzzy number) A is defined as a set of ordered pairs (x, x), where $x \in$ X and μ $x \times [0,1]$ has been dubbed the grade (or level) of membership of x in A in that representation. No other assumptions about x had been established at that point.

x is (or must be) a function of x, one assumed later. Originally, though, A was only a relation in the product space X [0, 1]. We understand that not every relationship has to be functional. It is simply a widely held belief that such a relationship between x and x should exist, resulting in a membership function.

$$\mu_A: X \rightarrow [0, 1]$$

With $\mu_x = \mu_A(x)$.

In our opinion such a point of view may be too restrictive and here most of the above and further quoted problems have their origin.

We summarize this in the form of the following primitive representations of a fuzzy number:

$$A = \{(x, y) \in \mathbb{R} \times [0, 1] : y = \mu_A(x),$$

Where μ_A function of membership or such a viewpoint, in our opinion, is overly restricted, and it is here that the majority of the difficulties mentioned above and elsewhere originate.

The following primitive representations of a fuzzy number are used to summarise this:

$$A = \{(x, y) \in \mathbb{R} \times [0, 1] : y$$

$$\mathbb{R}^+ \; - \; A \in \mathcal{FN}^+$$

Membership level of x we claimed that an extra feature of fuzzy number should be added in order to distinguish between two kinds of fuzzy numbers: Mirror images of positive numbers $A \in FN$, Positive real axis and negative numbers within the subset of fuzzy numbers FN corresponding to (quasi) convex membership functions supported.

$$B \in \mathcal{FN}^-$$

Defined directly on **R**. To solve this problem we have introduced in the concept of the orientation of the membership curve of **A**.

There have been several attempts to add non-standard operations on fuzzy numbers [1, 2, 5, 13, 14]. It was discovered that in order to design acceptable operations on fuzzy numbers, their membership functions must be invertible. In this manner, the *L-P* was proposed as well as the more general approach to fuzzy numbers regarded as fuzzy sets defined over the real axis, that fulfil some conditions, e.g. They're normal, compactly supported, and convex in various ways. The idea of representing fuzzy numbers using quasi-convex functions is addressed. In the class of quasi-convex functions, membership functions of fuzzy numbers were searched.

There the membership functions of fuzzy numbers were searched in the class of quasi-convex functions. Namely, if χr is a one-element set's $\{r\}$ with characteristic function, then we have specified two types of fuzzy number.

Check Your Progress

1. State about the arithmetic operations on intervals.
2. Define the term dependency problem.
3. What is arithmetic operations on fuzzy numbers?

4.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Interval arithmetic is a mathematical technique used to put bounds on rounding errors and measurement errors in mathematical computation.
2. The so-called dependency problem is a major obstacle to the application 1of interval arithmetic. Although interval methods can determine the range of elementary arithmetic operations and functions very accurately, this is not always true with more complicated functions.
3. There the membership functions of fuzzy numbers were searched in the class of quasi-convex functions. Namely, if χr is a one-element set's $\{r\}$ with $r \in \mathbb{R}$ characteristic function, then we have specified two types of fuzzy number.

4.5 SUMMARY

- Interval arithmetic (also known as interval mathematics, interval analysis, or interval computation) is a mathematical technique used to put bounds on rounding errors and measurement errors in mathematical computation.

NOTES

- In interval arithmetic, any variable x lies in the closed interval between a and b. A function f, when applied to x, yields an uncertain result; f produces an interval [c, d] which includes all the possible values for $f(x)$ for all $x \in [a, b]$
- The most common use is in software, to keep track of rounding errors in calculations and of uncertainties in the knowledge of the exact values of physical and technical parameters.
- The latter often arise from measurement errors and tolerances for components or due to limits on computational accuracy.
- The main objective of interval arithmetic is a simple way to calculate upper and lower bounds for the range of a function in one or more variables.
- An interval can also be defined as a locus of points at a given distance from the centre, clarification needed and this definition can be extended from real numbers to complex numbers.
- The basic algebraic operations for real interval numbers (real closed intervals) can be extended to complex numbers.
- Interval arithmetic can be extended, in an analogous manner, to other multidimensional number systems, such as quaternions and octonions, but with the expense that we have to sacrifice other useful properties of ordinary arithmetic.
- The methods of classical numerical analysis cannot be transferred one-to-one into interval-valued algorithms, as dependencies between numerical values are usually not taken into account.
- To work effectively in a real-life implementation, intervals must be compatible with floating point computing.
- The so-called dependency problem is a major obstacle to the application of interval arithmetic.
- Interval arithmetic can be used in various areas (such as set inversion, motion planning, set estimation or stability analysis) to treat estimates with no exact numerical value.
- Parameters for which no exact figures can be allocated often arise during the simulation of technical and physical processes. The production process of technical components allows certain tolerances, so some parameters fluctuate within intervals.

4.6 KEY WORDS

- **Interval arithmetic:** Interval arithmetic (also known as interval mathematics, interval analysis, or interval computation) is a mathematical technique used to put bounds on rounding errors and measurement errors in mathematical computation.

- **Multiple intervals:** Height and body weight both affect the value of the BMI (Body Mass Index). We have already treated weight as an uncertain measurement, but height is also subject to uncertainty. Height measurements in metres are usually rounded to the nearest centimetre: a recorded measurement of 1.79 metres actually means a height in the interval [1.785, 1.795].
- **Complex interval arithmetic:** An interval can also be defined as a locus of points at a given distance from the centre, clarification needed and this definition can be extended from real numbers to complex numbers.
- **Interval methods:** The methods of classical numerical analysis cannot be transferred one-to-one into interval-valued algorithms, as dependencies between numerical values are usually not taken into account.
- **Dependency problem:** The so-called dependency problem is a major obstacle to the application of interval arithmetic.

4.7 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. What do you understand by the arithmetic operations on intervals?
2. What is meant by arithmetic operations on fuzzy numbers?

Long-Answer Questions

1. Discuss briefly about the interval arithmetic giving appropriate examples.
2. Explain briefly about the operations on fuzzy numbers with the help of suitable examples.

4.8 FURTHER READINGS

Klir, George J. and Bo Yuan. 1995. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. New Jersey: Prentice Hall Inc.

Klir, George J. and Tina A. Folger. 2009. *Fuzzy Sets, Uncertainty and Information*. New Delhi: Prentice Hall of India.

Zimmermann, Hans-Jürgen. 1991. *Fuzzy Set Theory—And Its Applications*. New Delhi: Allied Publishers Private Limited.

BLOCK - II
FUZZY RELATIONS AND FUZZY MEASURES

NOTES

UNIT 5 ORDERING RELATIONS ON FUZZY SETS

Structure

5.0 INTRODUCTION

In mathematics, Sanchez was the first to examine the concept of fuzzy relational equations based on the max-min composition. He looked at the requirements and theories for resolving fuzzy relations on fuzzy sets, which are defined as mappings from sets to [0, 1]. Some theorems for existence and determination of solutions of certain basic fuzzy relation equations were given by him. However, he only gets the greatest element (or maximum solution) from the max-min (or min-max) composition of fuzzy relations. The work of Sanchez has put some light on this critical topic. Since then, a number of scholars have attempted to investigate the issue and offer solutions [1, 4, 10-12, 18, 34, 52, 75-80, 82, 108, 111]. Crisp relation characteristic functions can be generalised to allow for degrees of membership. A fuzzy set defined over the Cartesian product of crisp sets is referred to as a fuzzy relation. Membership functions can be used to define fuzzy relationships. The strength of the association between the items of the tuple is indicated by the membership grade.

Fuzzy compatibility relations, fuzzy ordering relations, fuzzy morphisms, and sup and inf compositions of fuzzy relations are all covered in this section.

Binary relations are mathematical functions that have been generalised. The primary distinction is that 1-Relations can allocate two or more elements from Y to each value of X.

In this unit, you will study about the fuzzy relation, binary fuzzy relations, fuzzy equivalence and similarity relations and fuzzy compatibility relations.

NOTES

5.1 OBJECTIVES

After going through this unit, you will be able to:

- Define fuzzy relation
- Understand about the basic concept of binary fuzzy relations
- Analyse the concept of fuzzy equivalence and similarity relations
- Explain the fuzzy compatibility relations

5.2 FUZZY RELATION

Sanchez was the first to examine the concept of fuzzy relational equations based on the max-min composition. He looked at the requirements and theories for resolving fuzzy relations on fuzzy sets, which are defined as mappings from sets to [0, 1]. Some theorems for existence and determination of solutions of certain basic fuzzy relation equations were given by him. However, he only gets the greatest element (or maximum solution) from the max-min (or min-max) composition of fuzzy relations. The work of Sanchez has put some light on this critical topic. Since then, a number of scholars have attempted to investigate the issue and offer solutions [1, 4, 10-12, 18, 34, 52, 75-80, 82, 108, 111].

The fuzzy relation equation is an equation of the form $A \cdot R = B$, where A and B are fuzzy sets, R is a fuzzy relation, and $A \cdot R$ stands for the composition of A with R. Crisp relation characteristic functions can be generalised to allow for degrees of membership. A fuzzy set defined over the Cartesian product of crisp sets is referred to as a fuzzy relation. Membership functions can be used to define fuzzy relationships. The strength of the association between the items of the tuple is indicated by the membership grade.

Then, let $X, Y \subseteq R$, be universal sets.

$$R = \{((x, y), \mu_R(x, y)) \mid (x, y) \in X \times Y\}$$

In $X \times Y \subseteq R$, this is referred to as a fuzzy relation.

Alternatively, if X and Y are two universal sets, the fuzzy relation $R(x, y)$ is written as,

$$R(x, y) = \left\{ \frac{\mu_R(x, y)}{(x, y)} \mid (x, y) \in X \times Y \right\}$$

Two-dimensional tables are frequently used to represent fuzzy relationships. A contented manner of entering the fuzzy relation R is represented by an $m \times n$ matrix.

NOTES

$$R = \begin{array}{c} \\ x_1 \\ \vdots \\ x_m \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & \cdots & y_n \end{array} \\ \begin{bmatrix} \mu_R(x_1, y_1) & \cdots & \mu_R(x_1, y_n) \\ \vdots & \ddots & \vdots \\ \mu_R(x_m, y_1) & \cdots & \mu_R(x_m, y_n) \end{bmatrix} \end{array}$$

Example 5.1:

Let $X = \{1, 2,3\}$ and $Y = \{1, 2\}$

If the membership function for each order pair (x, y) is given by, then

$$\mu_R(x, y) = e^{-(x-y)^2}$$

Come up with a fuzzy relationship.

Solution:

Using the common nomenclature, we have; the fuzzy relation may be defined in two ways.

$$R = \left\{ \frac{e^{-(1-1)^2}}{(1,1)}, \frac{e^{-(1-2)^2}}{(1,2)}, \frac{e^{-(2-1)^2}}{(2,1)}, \frac{e^{-(2-2)^2}}{(2,2)}, \frac{e^{-(3-1)^2}}{(3,1)}, \frac{e^{-(3-2)^2}}{(3,2)} \right\}$$

$$R = \left\{ \frac{1.0}{(1,1)}, \frac{0.37}{(1,2)}, \frac{0.37}{(2,1)}, \frac{1.0}{(2,2)}, \frac{0.02}{(3,1)}, \frac{0.37}{(3,2)} \right\}$$

The relational matrix is used in the second method, we have

$$R = \begin{bmatrix} 1 & 0.37 \\ 0.37 & 1 \\ 0.02 & 0.37 \end{bmatrix}$$

As a result, the membership function describes the degree of similarity between sets X and Y. Higher values in the relational matrix indicate a stronger relationship.

Fuzzy Relationship Projections

Let R be a fuzzily specified relation over $X_1 \times X_2 \times \ldots X_N$

Let, Y C E = $X_1, X_2, \ldots, X_N$ and E = $X_1, X_2, \ldots, X_N$.

Let, Y= $\{X_i, X_j\}$, $i, j \leq N$, $i \neq j$

Extensions of a Fuzzy Relationship in Cylindric Form: Cylindric extensions can be thought of as the inverse of projections. Let R be a fuzzy relation defined over all sets in Y's Cartesian product.

We'll now define the fuzzy equivalence relation. An equivalence relation is a reflexive, symmetric, and transitive crisp binary relation R (X, X). We can create a crisp set A_x for each element *x* in X, which contains all the items of X that are related to *x* via the equivalency relation.

$$A_x = \{y \mid (x, y) \in R\ (X, X)\}$$

NOTES

Because R is transitive and symmetric, each member of A_x is connected to all other members of A_x, indicating that A_x is certainly a subset of. Furthermore, no member of A_x is related to any X element that isn't part of A_x. He may desire to analyse the degree of similarity that the elements of X have to some defined $t_x \in X$ by referring to this set A_x as an equivalence class 17^{nt} to each other and only to each. For any $x \in X$, a similarity class can be represented as a fuzzy set in which the membership grade of any single element indicates R (X, X) with regard to x. Under the relation R, the members of each equivalence class can be considered equivalent *r*.

A partition on X is formed by the family of all such equivalence classes specified by the relation, which is generally denoted by X / R.

A fuzzy equivalency relation or similarity relation is a reflexive, symmetric, and transitive fuzzy binary connection. The following explanation on ideas can be applied to the alternative notion of fuzzy transitivity, while the max-min form of transitivity is assumed.

5.3 BINARY FUZZY RELATIONS

Binary relations are mathematical functions that have been generalised. The primary distinction is that 1-Relations can allocate two or more elements from Y to each value of X.

As a result, several basic operations on functions apply to binary relations as well.

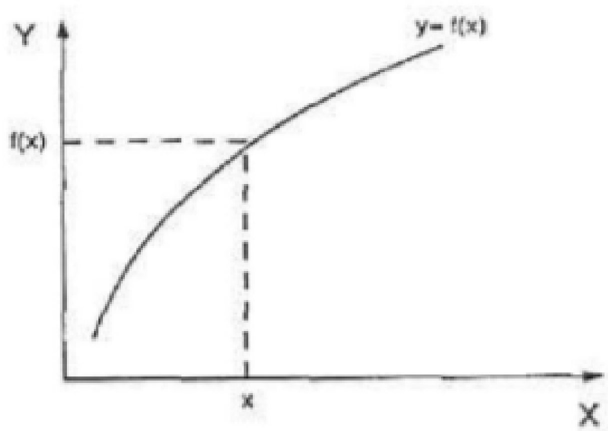

***Fig. 5.1** Binary Fuzzy Relation's Domain*

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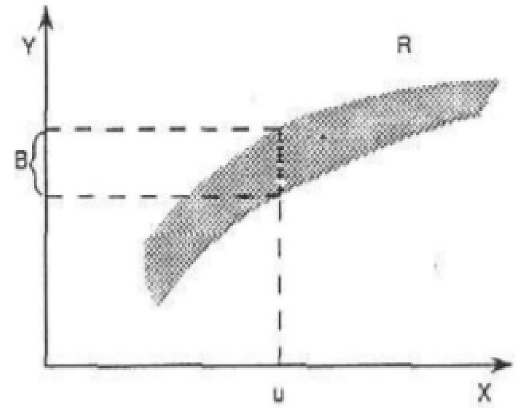

***Fig. 5.2** Pinnacle of a Binary Fuzzy Relationship*

A Binary Fuzzy Relation's Domain: The fuzzy set is the domain of a binary fuzzy relation R (X, Y).

$$Dom\ R(x) = \max R\ (x, y)$$

The Pinnacle of a Binary Fuzzy Relationship: $R\ (x, y)$ height is a number defined by

$$h(R) = \max R\ (x, y)$$

The largest membership grade in the relation is h(R).

The inverse of a fuzzy relation R is as follows:

$R^1(y, x) = R\ (x, y)$

Binary Fuzzy Relations Composition: We define the standard composition of two relations $R_1(X, Y)$ and $R_2(Y, Z)$ with a common set (Y) as:

$R\ (x, z) = [R_1, R_2]\ (x, z) = \max\ [\min\ (R_1(x, y)\ R_2(y, z)]$

A fuzzy subset μ of $X \times Y$ is a fuzzy binary relationship on X and Y. A fuzzy binary relation on the set X is hence a fuzzy subset of $X \times X$.

As this paper is concerned with only binary relations, so in the sequel we shall say relation instead of binary relation.

Binary Fuzzy Relation and their Properties

Binary relations are well-known to be generalised mathematical functions. Binary relations R (X, Y) can assign two or more elements of Y to each element of X, unlike functions from X to Y. Some basic functions operations, such as the inverse and composition, can also be used to binary relations.

The domain of a fuzzy relation R (X, Y) is a fuzzy set on X called dom R, and its membership function is defined by

$$\text{dom}\ (R(x)) = \max_{y \in Y} R\ (x, y)$$

$x \in X$ That is, each element of set X belongs to the domain of R to the degree that the strength of its strongest relation to any member of set Y is equivalent to the strength of its strongest relation to any member of set Y. R (X, Y) is a fuzzy relation on *Y*, with a membership function defined by

$$\text{ran } R(y) = \max_{x \in X} R(x, y)$$

That is, $h(R)$ is the highest membership grade a pair (x, y) can achieve in R.

5.4 FUZZY EQUIVALENCE AND SIMILARITY RELATIONS

We define (α, β)-fuzzy reflexive, symmetric, and transitive relations on a set X in this section. These definitions lay the way for (α, β) -fuzzy equivalence relations on the set X to be defined. We look at many sorts of (α, β) -fuzzy relations, specifically $(\in$, eve)-fuzzy relations**.**

A fuzzy subset of X × Y of the type $(x, y)\ r$

$$(x, y)_r\,(a, b) = \begin{cases} r & \text{if } x = a,\ y = b \\ 0 & \text{otherwise} \end{cases}$$

A fuzzy ordered pair is what it's called a fuzzy ordered pair this will be referred to as a fuzzy pair in the following sections.

Similarity Relations

The similarity relation is $R \subset X \times X$, which is reflexive, symmetric and transitive.

$$R = \begin{array}{c|cccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \hline x_1 & 1 & 0.2 & 1 & 0.6 & 0.2 & 0.6 \\ x_2 & 0.2 & 1 & 0.2 & 0.2 & 0.8 & 0.2 \\ x_3 & 1 & 0.2 & 1 & 0.6 & 0.2 & 0.6 \\ x_4 & 0.6 & 0.2 & 0.6 & 1 & 0.2 & 0.8 \\ x_5 & 0.2 & 0.8 & 0.2 & 0.2 & 1 & 0.2 \\ x_6 & 0.6 & 0.2 & 0.6 & 0.8 & 0.2 & 1 \end{array} \quad \text{is a similarity relation.}$$

Theorem 5.1

Each equivalence class R [X] is denoted by the

$$R[X] = \bigcup_{\alpha} \alpha / R_{\alpha}[X], \alpha \in [0,1]$$

Where R_{α} [X] denotes α-cut R $[\alpha]$

Definition

$A \subset X$, A is a fuzzy set, and its -cut, denoted by A_α, is a non-fuzzy set.

$$A_\alpha = \{x : \mu_A(x) \geq \alpha\}, \quad \alpha \in [0,1]$$

Example 5.2:

For $R[x_1]$ we have

$$R_{0.2}[x_1] = \{x_1, x_2, x_3, x_4, x_5, x_6\}$$

$$0.2/R_{0.2}[x_1] = 0.2/x_1 + 0.2/x_2 + 0.2/x_3 + 0.2/x_4 + 0.2/x_5 + 0.2/x_6$$

$$R_{0.6}[x_1] = \{x_1, x_3, x_4, x_6\}$$

$$0.6/R_{0.6}[x_1] = 0.6/x_1 + 0.6/x_3 + 0.6/x_4 + 0.6/x_6$$

$$R_1[x_1] = \{x_1, x_3\}$$

$$1/R_1[x_1] = 1/x_1 + 1/x_3$$

Class of equivalence for $R[x_1]$

$$R[x_1] = 0.2/x_1 + 0.2/x_2 + 0.2/x_3 + 0.2/x_4 + 0.2/x_5 + 0.2/x_6 + 0.6/x_1 + 0.6/x_3 + 0.6/x_4 + 0.6/x_6 + 1/x_1 + 1/x_3$$

$$R[x_1] = \max(0.2,0.6,1)/x_1 + 0.2/x_2 + \max(0.2,0.6,1)/x_3 + \max(0.2,0.6)/x_4 + 0.2/x_5 + \max(0.2,0.6)/x_6$$

$$R[x_1] = 1/x_1 + 0.2/x_2 + 1/x_3 + 0.6/x_4 + 0.2/x_5 + 0.6/x_6$$

Example 5.3:

For the similarity relation R, the equivalence class is

$$R[x_1] = 1/x_1 + 0.2/x_2 + 1/x_3 + 0.6/x_4 + 0.2/x_5 + 0.6/x_6$$

$$R[x_2] = 0.2/x_1 + 1/x_2 + 0.2/x_3 + 0.2/x_4 + 0.8/x_5 + 0.2/x_6$$

$$R[x_3] = 1/x_1 + 0.2/x_2 + 1/x_3 + 0.6/x_4 + 0.2/x_5 + 0.6/x_6$$

$$R[x_4] = 0.6/x_1 + 0.2/x_2 + 0.6/x_3 + 1/x_4 + 0.2/x_5 + 0.8/x_6$$

$$R[x_5] = 0.2/x_1 + 0.8/x_2 + 0.2/x_3 + 0.2/x_4 + 1/x_5 + 0.2/x_6$$

$$R[x_6] = 0.6/x_1 + 0.2/x_2 + 0.6/x_3 + 0.8/x_4 + 0.2/x_5 + 1/x_6$$

5.5 FUZZY COMPATIBILITY RELATIONS

NOTES

The definitions of fuzzy compatibility relations, fuzzy ordering relations, fuzzy morphisms, and sup and inf compositions of fuzzy relations are all covered in this section.

A one-to-one relationship a compatibility relation or tolerance relation is R (X, X) that is reflexive and symmetric. R (X, X) is frequently referred to as a proximity relation when it is a reflexive and symmetric fuzzy relation.

The term 'Compatibility' refers to a key idea in compatibility relationships. Classes of compatibility are a subset A of X such that (x, y) R for any x, y given a sharp compatibility relation R (X, X). A maximal compatibility class, also known as a maximum compatibility class, is a compatibility class that is not contained in any other compatibility class. A complete cover of X with regard to R is a family consisting of all the maximal compatibles induced by R on X.

Compass of a specific membership degree where R is a fuzzy compatibility relation. A subset A of X such that R (x, y) for every x, y A is a compatibility class. The concepts of maximal compatibles and complete cover are simple expansions of the notion of crisp compatibility relations.

In contrast to fuzzy cognitive maps, which are directed graphs, compatibility relations are frequently considered as reflexive undirected graphs. In this context, reflexivity means that each node in the graph has a loop connecting it to itself. Loops are typically omitted from visual representations of the graph, but they are assumed to exist. The relation-defined connections between nodes are not directed.

Each link is labelled with the value that corresponds to the membership grade R $(x, y) = R\ (x, y) = R\ (x, y) = R\ (x, y) = R\ (x, y)\ (y, x)$. The following example exemplifies this point.

Consider the following membership matrix for a fuzzy relation $R\ (X, X)$ defined on $X = x_1, x_2, ..., x_8$:

$$\begin{array}{cc} & \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{array} & \begin{bmatrix} 1 & .3 & 0 & 0 & .4 & 0 & 0 & .6 \\ .3 & 1 & .5 & .3 & 0 & 0 & 0 & 0 \\ 0 & .5 & 1 & 0 & 0 & .7 & .6 & .8 \\ 0 & .3 & 0 & 1 & .2 & 0 & .7 & .5 \\ .4 & 0 & 0 & .2 & 1 & 0 & 0 & 0 \\ 0 & 0 & .7 & 0 & 0 & 1 & .2 & 0 \\ 0 & 0 & .6 & .7 & 0 & .2 & 1 & .8 \\ .6 & 0 & .8 & .5 & 0 & 0 & .8 & 1 \end{bmatrix} \end{array}.$$

Because the matrix is symmetric and all elements on the major diagonal equal 1, the relationship represented is reflexive and symmetric, making it a

NOTES

compatibility relationship. Depicts the graph of the relationship, with complete-covers for > 0 and 0,.3,.1,.4,.6,.5,.2,.8,.7, 1. While similarity and compatibility relations are characterised by symmetry, Figure 5.3 depicts the graph of compatibility relation.

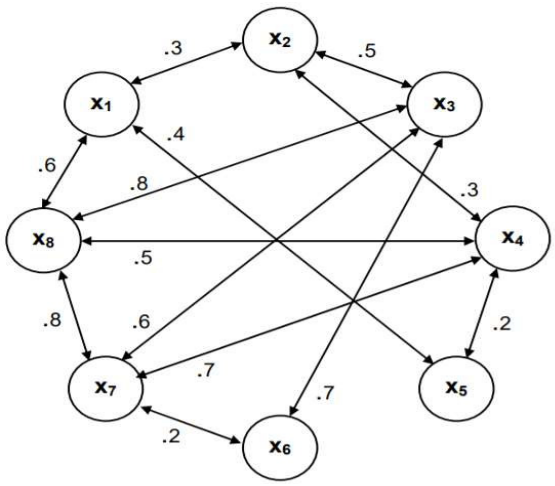

***Fig. 5.3** Depicts the Graph of Compatibility Relation*

A crisp binary relation R(X, X) that is reflexive, anti-symmetric and transitive is called the partial ordering. The common symbol ≤ is suggestive of the properties of this class of relations.

Thus $x \leq y$ denotes (x, y) ∈ R and signifies that x precedes y. The inverse partially ordering $R^{-1}(X, X)$ is suggested by the symbol ≥.

Check Your Progress

1. Give the definition of fuzzy relation.
2. What do you mean by the binary fuzzy relations?
3. Elaborate on the fuzzy equivalence.
4. What do you understand by similarity relations?
5. Describe the basic concept of fuzzy compatibility relations.

5.6 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. A fuzzy set defined over the Cartesian product of crisp sets is referred to as a fuzzy relation. Membership functions can be used to define fuzzy relationships. The strength of the association between the items of the tuple is indicated by the membership grade.

2. Binary relations are mathematical functions that have been generalised. The primary distinction is that 1-Relations can allocate two or more elements from Y to each value of X.

   A Binary Fuzzy Relation's Domain: The fuzzy set is the domain of a binary fuzzy relation R (X, Y).

   $$Dom\ R(x) = \max R\ (x, y)$$

3. The way for (α, β) -fuzzy equivalence relations on the set X to be defined. We look at many sorts of (α, β) -fuzzy relations, specifically ($\in$, eve)-fuzzy relations**.**

   A fuzzy subset of X × Y of the type (x, y) r

   $$(x, y)_r\ (a, b) = \begin{cases} r & \text{if } x = a,\ y = b \\ 0 & \text{otherwise} \end{cases}$$

   A fuzzy ordered pair is what it's called a fuzzy ordered pair this will be referred to as a fuzzy pair in the following sections.
4. The similarity relation is R $\subset$ X × X, which is reflexive, symmetric, and transitive.

   $$R = \begin{array}{c c} & \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} & \begin{bmatrix} 1 & 0.2 & 1 & 0.6 & 0.2 & 0.6 \\ 0.2 & 1 & 0.2 & 0.2 & 0.8 & 0.2 \\ 1 & 0.2 & 1 & 0.6 & 0.2 & 0.6 \\ 0.6 & 0.2 & 0.6 & 1 & 0.2 & 0.8 \\ 0.2 & 0.8 & 0.2 & 0.2 & 1 & 0.2 \\ 0.6 & 0.2 & 0.6 & 0.8 & 0.2 & 1 \end{bmatrix} \end{array}$$ is a similarity relation.

5. The definitions of fuzzy compatibility relations, fuzzy ordering relations, fuzzy morphisms, and sup and inf compositions of fuzzy relations are all covered in this section. A one-to-one relationship a compatibility relation or tolerance relation is R (X, X) that is reflexive and symmetric. R (X, X) is frequently referred to as a proximity relation when it is a reflexive and symmetric fuzzy relation.

5.7 SUMMARY

- Crisp relation characteristic functions can be generalised to allow for degrees of membership.
- A fuzzy set defined over the Cartesian product of crisp sets is referred to as a fuzzy relation.
- Membership functions can be used to define fuzzy relationships.
- The strength of the association between the items of the tuple is indicated by the membership grade.

NOTES

- Two-dimensional tables are frequently used to represent fuzzy relationships. A contented manner of entering the fuzzy relation *R* is represented by $m \times n$ matrix.
- Binary relations are mathematical functions that have been generalised. The primary distinction is that 1-Relations can allocate two or more elements from Y to each value of X.
- Binary relations are well-known to be generalised mathematical functions. Binary relations R (X, Y) can assign two or more elements of Y to each element of X, unlike functions from *X* to *Y*.
- The way for (α, β) -fuzzy equivalence relations on the set X to be defined. We look at many sorts of (α, β) -fuzzy relations, specifically $(\in$, eve)-fuzzy relations**.**
- The similarity relation is $R \subset X \times X$, which is reflexive, symmetric, and transitive.
- The definitions of fuzzy compatibility relations, fuzzy ordering relations, fuzzy morphisms, and sup and inf compositions of fuzzy relations are all covered in this section.
- A one-to-one relationship a compatibility relation or tolerance relation is R (X, X) that is reflexive and symmetric. R (X, X) is frequently referred to as a proximity relation when it is a reflexive and symmetric fuzzy relation.
- The term 'Compatibility' refers to a key idea in compatibility relationships.
- A maximal compatibility class, also known as a maximum compatibility class, is a compatibility class that is not contained in any other compatibility class.
- A complete cover of X with regard to R is a family consisting of all the maximal compatibles induced by R on X.

5.8 KEY WORDS

- **Fuzzy relation:** Crisp relation characteristic functions can be generalised to allow for degrees of membership. A fuzzy set defined over the Cartesian product of crisp sets is referred to as a fuzzy relation. Membership functions can be used to define fuzzy relationships. The strength of the association between the items of the tuple is indicated by the membership grade.
- **Binary fuzzy relations:** Binary relations are mathematical functions that have been generalised. The primary distinction is that 1-Relations can allocate two or more elements from Y to each value of X.
- **Fuzzy equivalence:** We define (α, β)-fuzzy reflexive, symmetric, and transitive relations on a set X in this section. These definitions lay the way for (α, β) -fuzzy equivalence relations on the set X to be defined. We look at many sorts of (α, β) -fuzzy relations, specifically $(\in$, eve)-fuzzy relations**.**

- **Similarity relations:** The similarity relation is $R \subset X \times X$, which is reflexive, symmetric and transitive.
- **Fuzzy compatibility relations:** The definitions of fuzzy compatibility relations, fuzzy ordering relations, fuzzy morphisms, and sup and inf compositions of fuzzy relations are all covered in this section. A one-to-one relationship a compatibility relation or tolerance relation is R (X, X) that is reflexive and symmetric. R (X, X) is frequently referred to as a proximity relation when it is a reflexive and symmetric fuzzy relation.

5.9 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. What do you mean by the fuzzy relation?
2. What do you understand by the term binary fuzzy relations?
3. Define fuzzy equivalence with appropriate example.
4. State about the similarity relations.
5. Explain the basic concept of fuzzy compatibility relations.

Long-Answer Questions

1. What are fuzzy relation? Explain the two-dimension table for fuzzy relation with the help of suitable examples.
2. What are binary fuzzy relations? Discuss the types of binary fuzzy relations giving examples of each type.
3. Briefly discuss the significance of fuzzy equivalence in fuzzy relation.
4. Discuss briefly about the similarity relations giving appropriate examples.
5. Explain about the fuzzy compatibility relations with the help of suitable examples.

5.10 FURTHER READINGS

Klir, George J. and Bo Yuan. 1995. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. New Jersey: Prentice Hall Inc.

Klir, George J. and Tina A. Folger. 2009. *Fuzzy Sets, Uncertainty and Information*. New Delhi: Prentice Hall of India.

Zimmermann, Hans-Jürgen. 1991. *Fuzzy Set Theory—And Its Applications*. New Delhi: Allied Publishers Private Limited.

UNIT 6 FUZZY MORPHISM

NOTES

Structure

6.0 INTRODUCTION

In mathematics, the concept of fuzzy order was introduced by generalizing the notion of reflexivity, antisymmetric and transitivity. Introduced by generalizing the notion of reûexivity, antisymmetric and transitivity. An order relation is a generalization of both set inclusion and the order relation on real line.

A transitive fuzzy relation is referred to as a fuzzy ordering. A fuzzy partial ordering, P, in particular, is a fuzzy ordering that is reflexive and antisymmetric, i.e., ($\mu P(x,y) > 0$ and $x \neq y$) $\mu P(y,x) = 0$. A fuzzy partial ordering in which $x \neq y \Rightarrow \mu s(x,y) > 0$ or $\mu s(y, x) > 0$ is called a fuzzy linear ordering.

As fuzzy algebraic structures and as specific fuzzy ordered sets. Conditions for transferring one concept to another are laid out in the same publication. A fuzzification of the carrier yields a fuzzy lattice as a fuzzy algebraic structure, whereas a fuzzification of the ordering relation yields a fuzzy lattice as a fuzzy ordered structure.

In 'Fuzzy Morphism Between Graphs' (GFM) is defined broadly, with standard graph-related problem descriptions as sub-cases (such as Graph and Subgraph Isomorphism). The GFM employs two fuzzy relations, one at the vertices and the other at the edges. The second is the concept of edge morphism, which is a relatively recent concept.

If two crisp binary relation R(x, x) and Q(y, y) are defined on set X, Y respectively.

Then a function $h: X \rightarrow Y$ is said to be homomorphism from (x, R) to (y, Q) if $(x_1, x_2) \in R = ((h(x_1), h(x_2)) \in Q$

In this unit, you will study about the fuzzy ordering relation and fuzzy morphisms.

6.1 OBJECTIVES

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After going through this unit, you will be able to:

- Describe what fuzzy ordering relation is
- Understand the fuzzy order
- Define the fuzzy morphisms
- Discuss about the category theory

6.2 FUZZY ORDERING RELATION

A transitive fuzzy relation is referred to as a fuzzy ordering. A fuzzy partial ordering, P, in particular, is a fuzzy ordering that is reflexive and antisymmetric, i.e., ($\mu P(x,y) > 0$ and $x \neq y$) $\mu P(y,x) = 0$. A fuzzy partial ordering in which $x \neq y \Rightarrow \mu s(x,y) > 0$ or $\mu s(y,x) > 0$ is called a fuzzy linear ordering.

The current study is based on fuzzy ordering relations theory. As lattice valued structures, these are examined. As a result, both the framework and the subject fall under the order theory. The fuzzy structures provided here provide cutworthy qualities, which means that crisp fuzzified qualities are kept when cut structures are used. As a result, all mappings' co-domains (fuzzy sets) are full lattices with no further operations. Specifically, such lattices allow crisp qualities to be transferred to cuts.

We employ well-known concepts from classical ordered structure theory.

A poset is a nonempty set P with an ordering relation (order), which is not always linear. It is commonly understood that by definition, an order on a given set satisfies the qualities of reflexivity, antisymmetry, and transitivity.

We also employ the concept of a weak ordering relation on a set P, which is an antisymmetric and transitive relation that also satisfies the weak reflexivity on P: for any x, y, ρ.

If $x\rho y$ is true, then $x\rho x$ and $y\rho y$ are true.

L is a complete lattice throughout this, with the top element and the bottom element; these are sometimes denoted as 1_L and 0_L. An atom in L is an element

$a \in L$ distinct from 0 if it covers the least element 0, i.e., if $0 \leq x \leq a$ implies $x=0$ or $x = a$. A monolith is defined as a single atom in L that is lower than all other non-zero elements in L. A lattice L is atomic if there is an atom a such that an x for every non-zero element x L. Finally, if every non-zero element is a join (supremum) of atoms, a lattice L is atomically created.

On a nonempty set, a fuzzy set S represents a mapping $\mu: S \rightarrow L$. For every $p \in L$, a cut set of is defined as the subset μp of S, such that $x \in \mu p$ if and only if $\mu(x) \geq p$ in L.

NOTES

A strong cut of μ is a subset of S, $\mu > p$ such that $x > p$ for any $p \in L$, $x \in \mu > p$ if and only if $\mu(x) > p$ in L.

If a property or an idea is transferred or generalized to fuzzy structures and is preserved by all cuts of the related fuzzy structure, it is said to be cutworthy.

Any mapping: X^2 L is an L–valued (lattice valued) relation on X, if X is a nonempty set. A cut relation is a cut of the associated fuzzy set, since a fuzzy relation is a fuzzy subset of X^2: for $p \in L$, a p-cut of is a subset p_p of X^2, such that $p_p = (x, y) \mid (x, y)$ p.

Now the key characteristics of fuzzy ordering relations. We use the following definition of fuzzy orders, which we believe is the most obvious lattice-valued expansion of the crisp order. This viewpoint is based on the fact that all non-trivial cuts of such fuzzy orders are crisp ordering relations, implying that their significance should be regarded in the cutworthy sense.

- If $(x, x) = 1$ for all $x \in X$, *a* L–valued relation is - reflexive.
- If $\rho(x, x) \geq \rho(x, y)$ and $\rho(x, x) \geq \rho(y,x)$, for any $x, y \in X$, it is weakly reflexive.
- $\rho(x, y) \wedge \rho(y, x) = 0$, for any $x, y \in X$, $x \neq y$, it is antisymmetric.
- $\rho(x, y) \wedge \rho(y, z) \leq \rho(x, z)$, for any $x, y, z \in X$, it is transitive.

If a L–valued relation on X is reflexive, antisymmetric, and transitive, it is a

L–fuzzy ordering relation (fuzzy order) on X. If an L–valued relation on X is weakly reflexive, antisymmetric, and transitive, it is a weak L–fuzzy ordering relation (weak fuzzy order). Observe that, just as in the crisp example, any fuzzy order is also a fuzzy weak order on the same set, whereas the reverse does not have to be true.

Theorem 6.1: If and only if all cuts except 0-cut are ordering relations, a relation Á: $S^2 \rightarrow L$ is an L-fuzzy ordering relation.

Proof: Let ρ: $S^2 \rightarrow L$ be a fuzzy ordering relation, and let $p \in L$ be the order if $p = 0$, then $\rho_0 = S^2$, and unless $|S| = 1$, it is not an ordering relation.

Let $p \neq 0$. Then, $(x, x) \in \rho_p$, by the reflexivity of ρ. If $(x, y) \in \rho_p$ and $(y, x) \in \rho_p$, then $\rho(x, y) \geq p$ and $\rho(y, x) \geq p$ and hence $\rho(x, y) \wedge \rho(y, x) \geq p$, which is true only in case $x = y$. If $(x, y) \in \rho_p$ and $(y, z) \in \rho_p$ then $\rho(x, y) \geq p$ and $\rho(y, z) \geq p$. Therefore, $\rho(x, z) \geq \rho(x, y) \wedge \rho(y, z) \geq p$ and hence $(x, z) \in \rho_p$.

Consider the case where Á is a fuzzy connection and all cuts (save the 0-cut) are ordering relations.

$\rho(x, x) = p \in L$ $p_p(x, x) = 1$ since all p cuts are reflexive relations.

Allow x to equal y and (x, y) (y, x*)* to equal p. Then we have $(x, y) \in p_p$ and

$(y, x) \in p_p$, and we get $p = 0$ because of the antisymmetry of p.

Similarly, we may show that ρ is transitive.

Fuzzy Orders

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Fuzzy Lattices are Studied from Two Perspectives: As fuzzy algebraic structures and as specific fuzzy ordered sets. Conditions for transferring one concept to another are laid out in the same publication. A fuzzification of the carrier yields a fuzzy lattice as a fuzzy algebraic structure, whereas a fuzzification of the ordering relation yields a fuzzy lattice as a fuzzy ordered structure.

We will create analog connections for fuzzy orders in this section.

On the other hand, we fuzzify a relation as a mapping from $P^2 \to \{0, 1\}$ starting with the same poset $(P, \leq)$. As a weakly reflexive, antisymmetric, and transitive mapping from P^2 to L, we establish a fuzzy weak ordering relation on P. It is simple to verify that the fuzzy weak order cut sets on P are crisp weak orderings on P. The following theorems establish a link between two fuzzifications

(a fuzzy poset and a fuzzy weak order). We will create analog connections for fuzzy orders in this section.

Lemma 6.1: Let L be a full lattice with a monolith a, and let $\rho: S^2 \to L$ be an L-fuzzy ordering relation. The strong cut Á > 0 is then a neat ordering relation on the set S.

Proof: We know that $(x, x) \in \rho > 0$ because $(x, x) = 1 > 0$. If $\rho(x, y) > 0$ and $\rho(y, x) > 0$, we have $\rho(x, y)$ a and $\rho(x, y) \geq a$, implying that $\in(x, y) \geq a$ and $\rho(x, y) \geq a$, and hence $x = y$.

If $\rho(x, y)$ and $\rho(y, z)$ are both greater than zero, we have $\rho(x, y) \wedge \rho(y, z) \geq a$, and hence, via fuzzy transitivity, $\rho(x, z) \geq a$. As a result, $e(x, z) \in \rho > 0$.

Remark: The assumption that L has a monolith is here essential. Indeed, the strong 0 cut in a fuzzy ordered set with the co-domain being a complete lattice without the unique atom need not be an ordered set.

Theorem 6.2. Assume that $\mu: P \to L$ is a L–fuzzy poset. Then there's the mapping $\rho: P_2 \to L$, which is defined by

$$\rho(x, y) = \begin{cases} 1, & if\ x = y \\ \mu(x) \wedge \mu(y), & if\ x < y \\ 0, & otherwise. \end{cases}$$

is a relation of L–fuzzy ordering.

Proof: By definition, the mapping ρ is reflexive. Assume $x = y$ to demonstrate the antisymmetry. If $x < y$ is <, not $y < x$, then it is not $y < x$ and vice versa, according to the antisymmetry of As a result, at least one of the numbers $\rho(x, y)$ or $\rho(y, x)$ is 0, and $\rho(x, y) \wedge \rho(y, x) = 0$.

6.3 FUZZY MORPHISMS

NOTES

The term 'Fuzzy Morphism Between Graphs' (GFM) is defined broadly, with standard graph-related problem descriptions as sub-cases (such as graph and subgraph isomorphism). The GFM employs two fuzzy relations, one at the vertices and the other at the edges. The second is the concept of edge morphism, which is a relatively recent concept.

If two crisp binary relation R(x, x) and Q(y, y) are defined on set X, Y respectively.

Then a function h: $X \rightarrow Y$ is said to be homomorphism from (x, R) to (y, Q) if $(x_1, x_2) \in R = ((h(x_1), h(x_2)) \in Q$

In otherwords, a homomorphism implies that for every two elements of set X which are related under their relation R their homomorphism images $h(x_1)$, $h(x_2)$ in the set Yare related under the relation Q.

If relations R(x, x) and Q(y, y) are furry then the criteria has a many to one function h must satisfy in order to be a strong homomorphism.

If h: $x \rightarrow y$ is a homomorphism from (x, R) to (y, Q) and is h is satisfy one-one, onto then v_0 is called an isomorphism.

If $y \subseteq x$ then h is called endomorphism.

A function isomorphism and endomorphism is called automorphism.

Definition

A category C consists of two classes, one of objects and the other of morphisms. There are two objects that are associated to every morphism, the source and the target. A morphism f with source X and target Y is written as f: $X \rightarrow Y$, and is represented diagrammatically by an arrow from X to Y.

For many common categories, objects are sets (often with some additional structure) and morphisms are functions from an object to another object. Therefore, the source and the target of a morphism are often called domain and codomain respectively.

Morphisms are equipped with a partial binary operation, called composition. The composition of two morphisms f and g is defined precisely when the target of f is the source of g, and is denoted $g \circ f$ (or sometimes simply gf). The source of $g \circ f$ is the source of f and the target of $g \circ f$ is the target of g. The composition satisfies two axioms:

For every object *X*, there exists a morphism $id_X : X \rightarrow X$ called the identity morphism on *X*, such that for every morphism $f : A \rightarrow B$

We have,

$id_B \circ f = f = f \circ id_A$.

Associativity

h ° (g ° f) = (h ° g) ° f whenever all the compositions are defined, i.e. when the target of f is the source of g, and the target of g is the source of h.

NOTES

For a concrete category (a category in which the objects are sets, possibly with additional structure, and the morphisms are structure-preserving functions), the identity morphism is just the identity function, and composition is just ordinary composition of functions.

The composition of morphisms is often represented by a commutative diagram. For example,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g \circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

The collection of all morphisms from X to Y is denoted HomC(X,Y) or simply Hom(X, Y) and called the hom-set between X and Y. Note that the term hom-set is something of a misnomer, as the collection of morphisms is not required to be a set. A category where Hom(X, Y) is a set for all objects X and Y is called locally small.

Note that the domain and codomain are in fact part of the information determining a morphism. For example, in the category of sets, where morphisms are functions, two functions may be identical as sets of ordered pairs (may have the same range), while having different codomains. The two functions are distinct from the viewpoint of category theory. If this disjointness does not hold, it can be assured by appending the domain and codomain to the morphisms.

Some Special Morphisms

Monomorphisms and Epimorphisms

A morphism f: $X \to Y$ is called a monomorphism if $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$ for all morphisms $g_1, g_2: Z \to X$. A monomorphism can be called a mono for short, and we can use monic as an adjective. A morphism f has a left inverse or is a split monomorphism if there is a morphism g: $Y \to X$ such that

$g \circ f = id_X$. Thus $f \circ g: Y \to Y$ is idempotent; that is, $(f \circ g)^2 = f \circ (g \circ f) \circ g = f \circ g$. The left inverse g is also called a retraction of f.

Morphisms with left inverses are always monomorphisms, but the converse is not true in general; a monomorphism may fail to have a left inverse. In concrete categories, a function that has a left inverse is injective. Thus in concrete categories, monomorphisms are often, but not always, injective. The condition of being an injection is stronger than that of being a monomorphism, but weaker than that of being a split monomorphism.

NOTES

Dually to monomorphisms, a morphism $f: X \rightarrow Y$ is called an epimorphism if $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$ for all morphisms $g_1, g_2: Y \rightarrow Z$. An epimorphism can be called an epic for short, and we can use epic as an adjective. A morphism f has a right inverse or is a split epimorphism if there is a morphism $g: Y \rightarrow X$ such that $f \circ g = id_Y$. The right inverse g is also called a section of f. Morphisms having a right inverse are always epimorphisms, but the converse is not true in general, as an epimorphism may fail to have a right inverse.

If a monomorphism f splits with left inverse g, then g is a split epimorphism with right inverse f. In concrete categories, a function that has a right inverse is surjective. Thus in concrete categories, epimorphisms are often, but not always, surjective. The condition of being a surjection is stronger than that of being an epimorphism, but weaker than that of being a split epimorphism. In the category of sets, the statement that every surjection has a section is equivalent to the axiom of choice.

A morphism that is both an epimorphism and a monomorphism is called a bimorphism.

Isomorphisms

A morphism $f: X \rightarrow Y$ is called an isomorphism if there exists a morphism $g: Y \rightarrow X$ such that $f \circ g = id_Y$ and $g \rightarrow f = id_X$. If a morphism has both left-inverse and right-inverse, then the two inverses are equal, so f is an isomorphism, and *g* is called simply the inverse of f. Inverse morphisms, if they exist, are unique. The inverse *g* is also an isomorphism, with inverse *f*. Two objects with an isomorphism between them are said to be isomorphic or equivalent.

While every isomorphism is a bimorphism, a bimorphism is not necessarily an isomorphism. For example, in the category of commutative rings the inclusion $Z \rightarrow Q$ is a bimorphism that is not an isomorphism. However, any morphism that is both an epimorphism and a split monomorphism, or both a monomorphism and a split epimorphism, must be an isomorphism. A category, such as Set, in which every bimorphism is an isomorphism is known as a balanced category.

Endomorphisms and Automorphisms

A morphism $f: X \rightarrow X$ (that is, a morphism with identical source and target) is an endomorphism of X. A split endomorphism is an idempotent endomorphism f if f admits a decomposition $f = h \circ g$ with $g \circ h = id$. In particular, the Karoubi envelope of a category splits every idempotent morphism.

An automorphism is a morphism that is both an endomorphism and an isomorphism. In every category, the automorphisms of an object always form a group, called the automorphism group of the object.

Category Theory

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Category theory formalizes mathematical structure and its concepts in terms of a labeled directed graph called a category, whose nodes are called objects, and whose labelled directed edges are called arrows (or morphisms). A category has two basic properties: the ability to compose the arrows associatively, and the existence of an identity arrow for each object. The language of category theory has been used to formalize concepts of other high-level abstractions, such as sets, rings and groups. Informally, category theory is a general theory of functions.

Several terms used in category theory, including the term 'Morphism' are used differently from their uses in the rest of mathematics. In category theory, morphisms obey conditions specific to category theory itself.

Samuel Eilenberg and Saunders Mac Lane introduced the concepts of categories, functors, and natural transformations from 1942–45 in their study of algebraic topology, with the goal of understanding the processes that preserve mathematical structure.

Category theory has practical applications in programming language theory, for example the usage of monads in functional programming. It may also be used as an axiomatic foundation for mathematics, as an alternative to set theory and other proposed foundations.

Basic Concepts

Categories represent abstractions of other mathematical concepts. Many areas of mathematics can be formalised by category theory as categories. Hence category theory uses abstraction to make it possible to state and prove many intricate and subtle mathematical results in these fields in a much simpler way.

A basic example of a category is the category of sets, where the objects are sets and the arrows are functions from one set to another. However, the objects of a category need not be sets, and the arrows need not be functions. Any way of formalising a mathematical concept such that it meets the basic conditions on the behaviour of objects and arrows is a valid category—and all the results of category theory apply to it.

The 'Arrows' of category theory are often said to represent a process connecting two objects, or in many cases a 'Structure-Preserving' transformation connecting two objects. There are, however, many applications where much more abstract concepts are represented by objects and morphisms. The most important property of the arrows is that they can be 'Composed', in other words, arranged in a sequence to form a new arrow.

Applications of Categories

Categories now appear in many branches of mathematics, some areas of theoretical computer science where they can correspond to types or to database schemas, and mathematical physics where they can be used to describe vector spaces. Probably the first application of category theory outside pure mathematics was the 'Metabolism-Repair' model of autonomous living organisms by Robert Rosen.

Utility

The study of categories is an attempt to axiomatically capture what is commonly found in various classes of related mathematical structures by relating them to the structure-preserving functions between them. A systematic study of category theory then allows us to prove general results about any of these types of mathematical structures from the axioms of a category.

Consider the following example. The class Grp (Gr or class of all groups) of groups consists of all objects having a 'Group Structure'. One can proceed to prove theorems about groups by making logical deductions from the set of axioms defining groups. For example, it is immediately proven from the axioms that the identity element of a group is unique.

Instead of focusing merely on the individual objects (e.g., Groups) possessing a given structure, category theory emphasizes the morphisms – the structure-preserving mappings – between these objects; by studying these morphisms, one is able to learn more about the structure of the objects. In the case of groups, the morphisms are the group homomorphisms. A group homomorphism between two groups 'Preserves the Group Structure' in a precise sense; Informally it is a 'Process' taking one group to another, in a way that carries along information about the structure of the first group into the second group. The study of group homomorphisms then provides a tool for studying general properties of groups and consequences of the group axioms.

A similar type of investigation occurs in many mathematical theories, such as the study of continuous maps (morphisms) between topological spaces in topology (the associated category is called top), and the study of smooth functions (morphisms) in manifold theory.

Not all categories arise as 'Structure Preserving (Set) Functions', however; the standard example is the category of homotopies between pointed topological spaces.

If one axiomatizes relations instead of functions, one obtains the theory of allegories.

Functors

A category is itself a type of mathematical structure, so we can look for 'Processes' which preserve this structure in some sense; such a process is called a functor.

Diagram chasing is a visual method of arguing with abstract 'Arrows Joined in Diagrams'. Functors are represented by arrows between categories, subject to specific defining commutativity conditions. Functors can define (construct) categorical diagrams and sequences (cf. Mitchell, 1965).

NOTES

A functor associates to every object of one category an object of another category, and to every morphism in the first category a morphism in the second.

As a result, this defines a category of categories and functors – the objects are categories, and the morphisms (between categories) are functors.

Studying categories and functors is not just studying a class of mathematical structures and the morphisms between them but rather the relationships between various classes of mathematical structures. This fundamental idea first surfaced in algebraic topology. Difficult topological questions can be translated into algebraic questions which are often easier to solve. Basic constructions, such as the fundamental group or the fundamental groupoid of a topological space, can be expressed as functors to the category of groupoids in this way, and the concept is pervasive in algebra and its applications.

Natural Transformations

Abstracting yet again, some diagrammatic and/or sequential constructions are often 'Naturally Related' – a vague notion, at first sight. This leads to the clarifying concept of natural transformation, a way to 'Map' one functor to another. Many important constructions in mathematics can be studied in this context. 'Naturality' is a principle, like general covariance in physics that cuts deeper than is initially apparent. An arrow between two functors is a natural transformation when it is subject to certain naturality or commutativity conditions.

Functors and natural transformations (Naturality) are the key concepts in category theory.

Check Your Progress

1. What do you understand by the fuzzy ordering relation?
2. Define the term fuzzy order.
3. Explain the term fuzzy morphism.
4. What is category theory?

6.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. A transitive fuzzy relation is referred to as a fuzzy ordering. A fuzzy partial ordering, P, in particular, is a fuzzy ordering that is reflexive and antisymmetric, i.e., ($\mu P(x,y) > 0$ and $x \neq y$) $\mu P(y,x) = 0$. A fuzzy partial ordering in which $x \neq y \Rightarrow \mu s(x,y) > 0$ or $\mu s(y,x) > 0$ is called a fuzzy linear ordering.
2. Fuzzy lattices are studied from two perspectives: as fuzzy algebraic structures and as specific fuzzy ordered sets. Conditions for transferring one concept

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to another are laid out in the same publication. A fuzzification of the carrier yields a fuzzy lattice as a fuzzy algebraic structure, whereas a fuzzification of the ordering relation yields a fuzzy lattice as a fuzzy ordered structure.

3. The term 'Fuzzy Morphism between Graphs' (GFM) is defined broadly, with standard graph-related problem descriptions as sub-cases (such as graph and subgraph isomorphism). The GFM employs two fuzzy relations, one at the vertices and the other at the edges. The second is the concept of edge morphism, which is a relatively recent concept.

   If two crisp binary relation R(x, x) and Q(y, y) are defined on set X, Y respectively. Then a function h: $X \rightarrow Y$ is said to be homomorphism from (x, R) to (y, Q) if

   $(x_1, x_2) \in R = ((h(x_1), h(x_2)) \in Q$

4. Category theory formalizes mathematical structure and its concepts in terms of a labeled directed graph called a category, whose nodes are called objects, and whose labelled directed edges are called arrows (or morphisms).

6.5 SUMMARY

- A transitive fuzzy relation is referred to as a fuzzy ordering. A fuzzy partial ordering, P, in particular, is a fuzzy ordering that is reflexive and antisymmetric, i.e., ($\mu P(x,y) > 0$ and $x \neq y$) $\mu P(y,x) = 0$. A fuzzy partial ordering in which $x \neq y \Rightarrow \mu s(x,y) > 0$ or $\mu s(y,x) > 0$ is called a fuzzy linear ordering.
- As lattice valued structures, these are examined. As a result, both the framework and the subject fall under the order theory.
- The fuzzy structures provided here provide cutworthy qualities, which means that crisp fuzzified qualities are kept when cut structures are used.
- If a property or an idea is transferred or generalized to fuzzy structures and is preserved by all cuts of the related fuzzy structure, it is said to be cutworthy.
- This viewpoint is based on the fact that all non-trivial cuts of such fuzzy orders are crisp ordering relations, implying that their significance should be regarded in the cutworthy sense.
- If a L–valued relation on X is reflexive, antisymmetric, and transitive, it is a L–fuzzy ordering relation (fuzzy order) on X. If an L–valued relation on X is weakly reflexive, antisymmetric, and transitive, it is a weak L–fuzzy ordering relation (weak fuzzy order).

- As fuzzy algebraic structures and as specific fuzzy ordered sets.
- A fuzzification of the carrier yields a fuzzy lattice as a fuzzy algebraic structure, whereas a fuzzification of the ordering relation yields a fuzzy lattice as a fuzzy ordered structure.
- The term 'Fuzzy Morphism Between Graphs' (GFM) is defined broadly, with standard graph-related problem descriptions as sub-cases (Such as Graph and Subgraph Isomorphism). The GFM employs two fuzzy relations, one at the vertices and the other at the edges. The second is the concept of edge morphism, which is a relatively recent concept.
- If two crisp binary relation R(x, x) and Q(y, y) are defined on set X, Y respectively. Then a function h: $X \to Y$ is said to be homomorphism from (x, R) to (y, Q) if $(x_1, x_2) \in R = ((h(x_1), h(x_2)) \in Q$
- A homomorphism implies that for every two elements of set X which are related under their relation R their homomorphism images $h(x_1)$, $h(x_2)$ in the set Yare related under the relation Q.
- If relations R(x, x) and Q(y, y) are furry then the criteria has a many to one function h must satisfy in order to be a strong homomorphism.
- If h: $x \to y$ is a homomorphism from (x, R) to (y, Q) and is h is satisfy one-one, onto then v_0 is called an isomorphism.
- If $y \subseteq x$ then h is called endomorphism.
- A function isomorphism and endomorphism is called automorphism.
- A category C consists of two classes, one of objects and the other of morphisms. There are two objects that are associated to every morphism, the source and the target. A morphism f with source X and target Y is written f: $X \to Y$, and is represented diagrammatically by an arrow from X to Y.
- A morphism f: $X \to Y$ is called a monomorphism if $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$ for all morphisms $g_1, g_2: Z \to X$. A monomorphism can be called a mono for short, and we can use monic as an adjective. A morphism f has a left inverse or is a split monomorphism if there is a morphism g:$Y \to X$ such that $g \circ f = id_X$. Thus $f \circ g: Y \to Y$ is idempotent; that is, $(f \circ g)^2 = f \circ (g \circ f) \circ g = f \circ g$. The left inverse g is also called a retraction of f.
- If a morphism has both left-inverse and right-inverse, then the two inverses are equal, so f is an isomorphism, and *g* is called simply the inverse of f.
- A morphism f: $X \to X$ (that is, a morphism with identical source and target) is an endomorphism of X. A split endomorphism is an idempotent

endomorphism f if f admits a decomposition $f = h \circ g$ with $g \circ h = id$. In particular, the Karoubi envelope of a category splits every idempotent morphism.

NOTES

- Category theory formalizes mathematical structure and its concepts in terms of a labeled directed graph called a category, whose nodes are called objects, and whose labelled directed edges are called arrows (or morphisms).
- The 'Arrows' of category theory are often said to represent a process connecting two objects, or in many cases a 'Structure-Preserving' transformation connecting two objects.

6.6 KEY WORDS

- **Fuzzy ordering relation:** A transitive fuzzy relation is referred to as a fuzzy ordering. A fuzzy partial ordering, P, in particular, is a fuzzy ordering that is reflexive and antisymmetric, i.e., ($\mu P(x,y) > 0$ and $x \neq y$) $\mu P(y,x) = 0$. A fuzzy partial ordering in which $x \neq y \Rightarrow \mu s(x,y) > 0$ or $\mu s(y,x) > 0$ is called a fuzzy linear ordering.
- **Fuzzy orders:** Fuzzy lattices are studied from two perspectives: as fuzzy algebraic structures and as specific fuzzy ordered sets. Conditions for transferring one concept to another are laid out in the same publication. A fuzzification of the carrier yields a fuzzy lattice as a fuzzy algebraic structure, whereas a fuzzification of the ordering relation yields a fuzzy lattice as a fuzzy ordered structure.
- **Fuzzy morphisms:** The term 'Fuzzy Morphism between Graphs' (GFM) is defined broadly, with standard graph-related problem descriptions as sub-cases (such as graph and subgraph isomorphism). The GFM employs two fuzzy relations, one at the vertices and the other at the edges. The second is the concept of edge morphism, which is a relatively recent concept.
- **Isomorphisms:** A morphism $f: X \rightarrow Y$ is called an isomorphism if there exists a morphism $g: Y \rightarrow X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$.
- **Automorphisms:** An automorphism is a morphism that is both an endomorphism and an isomorphism. In every category, the automorphisms of an object always form a group, called the automorphism group of the object.
- **Category theory:** Category theory formalizes mathematical structure and its concepts in terms of a labeled directed graph called a category, whose nodes are called objects, and whose labelled directed edges are called arrows (or morphisms).

6.7 SELF-ASSESSMENT QUESTIONS AND EXERCISES

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Short-Answer Questions

1. What do you understand by the fuzzy ordering relation?
2. Define the term fuzzy order.
3. What is fuzzy morphism?
4. Elaborate on the category theory.

Long-Answer Questions

1. Briefly define the fuzzy ordering relation with the help of appropriate examples.
2. What do you understand by the fuzzy order? Give examples.
3. Define the basic concept of fuzzy morphism with the help of appropriate examples.
4. Describe the classical definition of category theory with the help of examples.

6.8 FURTHER READINGS

Klir, George J. and Bo Yuan. 1995. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. New Jersey: Prentice Hall Inc.

Klir, George J. and Tina A. Folger. 2009. *Fuzzy Sets, Uncertainty and Information*. New Delhi: Prentice Hall of India.

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UNIT 7 FUZZY MEASURES

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Structure

7.0 INTRODUCTION

In mathematics, fuzzy measure theory considers generalized measures in which the additive property is replaced by the weaker property of monotonicity. The central concept of fuzzy measure theory is the fuzzy measure which was introduced by Choquet in 1953 and independently defined by Sugeno in 1974 in the context of fuzzy integrals. There exists a number of different classes of fuzzy measures including plausibility/belief measures; possibility/necessity measures; and probability measures which are a subset of classical measures.

Different ways for capturing partial, uncertain, and imprecise information are known as belief and plausibility measures in Dempster–Shafer Theory (DST) and fuzzy sets. The belief and plausibility measurements are then applied to IFSs, and Belief-Plausibility Intervals (BPIs) are created.

In mathematics, a probability measure is a real-valued function defined on a set of events in a probability space that satisfies measure properties, such as countable additivity. The difference between a probability measure and the more general notion of measure (which includes concepts like area or volume) is that a probability measure must assign value 1 to the entire probability space.

Intuitively, the additivity property says that the probability assigned to the union of two disjoint events by the measure should be the sum of the probabilities of the events, e.g. the value assigned to '1 or 2' in a throw of a die should be the sum of the values assigned to '1' and '2'.

Probability measures have applications in diverse fields, from physics to finance and biology.

In this unit, you will study about the fuzzy measure, belief and plausibility measures and probability measure.

7.1 OBJECTIVES

After going through this unit, you will be able to:

- Explain the fuzzy measure
- Understand the basics concepts of belief and plausibility measures
- Describe the concept of probability measure

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7.2 FUZZY MEASURE

Fuzzy measure theory considers generalized measures in which the additive property is replaced by the weaker property of monotonicity. The central concept of fuzzy measure theory is the fuzzy measure which was introduced by Choquet in 1953 and independently defined by Sugeno in 1974 in the context of fuzzy integrals. There exists a number of different classes of fuzzy measures including plausibility/ belief measures; possibility/necessity measures; and probability measures which are a subset of classical measures.

Definitions

Let X be a universe of discourse, be a class of subsets of X and E, F $\in$ C.

A function g: C $\rightarrow \mathbb{R}$

where,

1. $\emptyset \in \mathcal{C} \Rightarrow g(\emptyset) = 0$
2. $E \subseteq F \Rightarrow g(E) \leq g(F)$

called a fuzzy measure. A fuzzy measure is called normalized or regular if g (X) =1

Properties of Fuzzy Measures

A fuzzy measure is:

- Additive if for any $E, F \in C$ such that $E \cap F = \theta$, we have

  $g(E \cup F) = g(E) + g(F)$.;

- Supermodular if for any $E, F \in C$ we have

  $g(E \cup F) + g(E \cap F) \geq g(E) + g(F)$;

- Sub modular if for any $E, F \in C$ we have

  $g(E \cup F) + g(E \cap F) \leq g(E) + g(F)$;

- Superadditive if for any E, $F \in C$ such that $E \cap F = \theta$, we have $g(E \cup F) \geq g(E) + g(F)$;

NOTES

- Sub additive if for any E, $F \in C$ such that $E \cap F = \theta$, we have $g(E \cup F) \leq g(E) + g(F)$;
- Symmetric if for any E, $F \in C$ such that $|E| = |F|$ implies $g(E) = g(F)$
- Boolean if for any E, $F \in C$ such that $g(E) = 0$ or $g(E) = 1$

Understanding the properties of fuzzy measures is useful in application. When a fuzzy measure is used to define a function, such as the Sugeno integral or Choquet integral, these properties will be crucial in understanding the function's behaviour. For instance, the Choquet integral with respect to an additive fuzzy measure reduces to the Lebesgue integral. In discrete cases, a symmetric fuzzy measure will result in the Ordered Weighted Averaging (OWA) operator. Submodular fuzzy measures result in convex functions, while supermodular fuzzy measures result in concave functions when used to define a Choquet integral.

Möbius Representation

Let g be a fuzzy measure, the Möbius representation of g is given by the set function M, where for every E, F $\subseteq$ X,

$$M(E) = \sum_{F \subseteq E} (-1)^{|E \setminus F|} g(F).$$

The equivalent axioms in Möbius representation are:

1. $M(\emptyset) = 0$.
2. $\sum_{F \subseteq E | i \in F} M(F) \geq 0$, for all $E \subseteq \mathbf{X}$ and all $i \in E$

A fuzzy measure in Möbius representation M is called normalized if,

$$\sum_{E \subseteq \mathbf{X}} M(E) = 1.$$

Möbius representation can be used to give an indication of which subsets of X interact with one another. For instance, an additive fuzzy measure has Möbius values all equal to zero except for singletons. The fuzzy measure g in standard representation can be recovered from the Möbius form using the Zeta transform:

$$g(E) = \sum_{F \subseteq E} M(F), \forall E \subseteq \mathbf{X}.$$

Simplification Assumptions for Fuzzy Measures

Fuzzy measures are defined on a semi ring of sets or monotone class which may be as granular as the power set of X, and even in discrete cases the number of variables can as large as $2^{|X|}$. For this reason, in the context of multi-criteria decision analysis and other disciplines, simplification assumptions on the fuzzy measure have been introduced so that it is less computationally expensive to determine and use. For instance, when it is assumed the fuzzy measure is additive, it will hold that $g(E)=\sum_{i\in E} g(\{i\})$ and the values of the fuzzy measure can be evaluated from the values on X. Similarly, a symmetric fuzzy measure is defined uniquely by |X| values. Two important fuzzy measures that can be used are the Sugeno- or λ-fuzzy measure and k-additive measures, introduced by Sugeno and Grabisch respectively.

Sugeno λ-measure

The Sugeno λ -measure is a special case of fuzzy measures defined iteratively. It has the following definition:

Definition:

Let $\{x_1, ..., x_n\}$ be a finite set and let $\lambda \in \{-1, +\infty\}$ A Sugeno » -measure is a function g: 2^X [0, 1]

1. $g(X) = 1$.
2. if $A, B \subseteq \mathbf{X}$ (alternatively $A, B \in 2^{\mathbf{X}}$) with $A \cap B = \emptyset$ then $g(A \cup B) = g(A) + g(B) + \lambda g(A)g(B)$.

As a convention, the value of g at a singleton se $\{x_i\}$ is called a density and is denoted by $g_i = g(\{x_i\})$ In addition, we have that » satisfies the property

$$\lambda + 1 = \prod_{i=1}^{n}(1 + \lambda g_i).$$

Tahani and Keller as well as Wang and Klir have showed that once the densities are known, it is possible to use the previous polynomial to obtain the values of » uniquely.

***K*-Additive Fuzzy Measure**

The *k*-additive fuzzy measure limits the interaction between the subsets $E \subseteq X$ to size $|E| = k$. This drastically reduces the number of variables needed to define the fuzzy measure, and as *k* can be anything from 1 (in which case the fuzzy measure is additive) to X, it allows for a compromise between modelling ability and simplicity.

Definition

A discrete fuzzy measure g on a set X is called k-additive ($1 \le k \le |x|$ if its Möbius representation verifies M (E) = 0, whenever $E > k$ for any $E \subseteq X$ and there exists a subset F with *k* elements such that M (F) ≠ 0.

Shapley and Interaction Indices

In game theory, the Shapley value or Shapley index is used to indicate the weight of a game. Shapley values can be calculated for fuzzy measures in order to give some indication of the importance of each singleton. In the case of additive fuzzy measures, the Shapley value will be the same as each singleton.

For a given fuzzy measure g, and $|X| = n$, the Shapley index for every $i, \ldots n \in X$ is:

$$\phi(i) = \sum_{E \subseteq \mathbf{X} \setminus \{i\}} \frac{(n - |E| - 1)!|E|!}{n!} [g(E \cup \{i\}) - g(E)].$$

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The Shapley value is the vector $\phi(g) = (\psi(1), \ldots, \psi(n))$.

7.3 BELIEF AND PLAUSIBILITY MEASURES

There is a lot of random, fuzzy, unclear, imprecise and ambiguous uncertainty in the real world. Probability has traditionally been used to model uncertainty in the presence of unpredictability. Dempster–Shafer Theory (DST)-based belief and plausibility metrics have arisen as another approach of quantifying uncertainty, and they have been intensively explored and utilized in a variety of fields. Shafer interpreted the theory of evidence as an attempt to broaden probability theory. On the other hand, because two-value logic based on set theory is difficult to use when dealing with complicated systems, Zadeh proposed Fuzzy Sets (FSs) as an extension of regular sets. Since then, fuzziness has been widely used to deal with a different form of uncertainty than probability or randomness. The membership values of FSs range from 0 to 1. Zadeh was the first to present a generalization of DST to fuzzy sets. Following that, numerous DST generalizations to fuzzy sets were developed. Less generalization of DST to IFSs has been proposed.

Hwang and Yang suggested a fuzzy integral-based generalization of belief and plausibility functions to IFSs, which captures additional information on the shift in intuitionistic fuzzy focal elements on DST.

They stated that there is a close relation between IFS theory and DST evidence theory, allowing them to aggregate local criteria without restriction. They also proposed a method for resolving these issues based on the interpretation of IFSs in the context of DST. We provide Belief (Bel) and Plausibility (Pl) measurements on IFSs in the context of DST in this research.

In terms of degrees of membership and nonmembership in an IFS, we first provide a straightforward and intuitive technique to construct the degrees of belief and plausibility. Then, Intuitionistic Fuzzy Set (IFSs), we propose belief and plausibility measurements and construct Belief-Plausibility Intervals (BPIs). Measures of distance and similarity are useful for determining the degree of difference and similarity between two items. Many distance and similarity measurements between fuzzy sets IFSs have been extensively investigated and applied. Using the Hausdorff metric, we also establish new distance and similarity measurements between BPIs of IFSs in this paper. By developing a belief-plausibility TOPSIS (Technique for Order Performance by Similarity to Ideal Solution), these similarity measures between BPIs of IFSs are subsequently employed in multicriteria decision-making.

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Dempster–Shafer Theory

Dempster first introduced the lower and higher probability via a multivalued mapping. Shafer built belief and plausibility metrics on ordinary subsets based on Dempster's lower and upper probability a decade later.

Consider a probability space with a triplet (Ω, M, P) and a multivalued mapping ψ from Ω to Θ that assigns a subset ψ (ω) C Θ to every element $\omega \in \Omega$.

To put it another way, ψ is a set-valued function from Ω to the power set 2^{Θ} of Θ we have $S_* = \{\omega \in \Omega : \Psi(\omega) \subset S, \Psi(\omega) \neq \phi\}$

And $S^* = \{\omega \in \Omega : \Psi(\omega) \cap S \neq \phi\}$.

Specifically, $\Theta_* = \Theta^* = \Omega$.

Assume that K is the class of subsets S of Θ, and that S_* and S^* are members of it M.

The lower (P_*) and upper (P^*) probabilities of (S) in (K) are then defined as,

$$P_*(S) = \frac{P(S_*)}{P(\Theta^*)},$$

$$P^*(S) = \frac{P(S^*)}{P(\Theta^*)},$$

It's worth noting that if ψ is a single-valued function, ψ transforms into a random variable $P_*(S) = P^*(S)$.

If ψ is a multivalued mapping, on the other hand, an outcome ω may be mapped to several values.

7.4 PROBABILITY THEORY

Probability theory is the branch of mathematics concerned with probability. Although there are several different probability interpretations, probability theory treats the concept in a rigorous mathematical manner by expressing it through a set of axioms. Typically these axioms formalise probability in terms of a probability space, which assigns a measure taking values between 0 and 1, termed the probability measure, to a set of outcomes called the sample space. Any specified subset of these outcomes is called an event. Central subjects in probability theory include discrete and continuous random variables, probability distributions and stochastic processes, which provide mathematical abstractions of non-deterministic or uncertain processes or measured quantities that may either be single occurrences or evolve over time in a random fashion. Although it is not possible to perfectly predict random events, much can be said about their behaviour. Two major results in probability theory describing such behaviour are the law of large numbers and

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the central limit theorem. Although having a long history (with a first import theorem stated by Thomas Bayes), an axiomatization was done during the beginning and mid of the 20th century. During that time, the connection between probability and statistic was established, thereby also connecting to the field of measure theory. In its modern form, it developed mostly in parallel to more prominent physical theories (general relativity and quantum mechanics) and is often overshadowed by them, and therefore sometimes leading to inconsistent theories or wrong beliefs.

As a mathematical foundation for statistics, probability theory is essential to many human activities that involve quantitative analysis of data. Methods of probability theory also apply to descriptions of complex systems given only partial knowledge of their state, as in statistical mechanics or sequential estimation. A great discovery of twentieth-century physics was the probabilistic nature of physical phenomena at atomic scales, described in quantum mechanics.

Probability Measure

A probability measure is a real-valued function defined on a set of events in a probability space that satisfies measure properties, such as countable additively. The difference between a probability measure and the more general notion of measure (which includes concepts like area or volume) is that a probability measure must assign value 1 to the entire probability space.

Intuitively, the additively property says that the probability assigned to the union of two disjoint events by the measure should be the sum of the probabilities of the events, e.g., the value assigned to '1 or 2' in a throw of a die should be the sum of the values assigned to '1' and '2'.

Probability measures have applications in diverse fields, from physics to finance and biology.

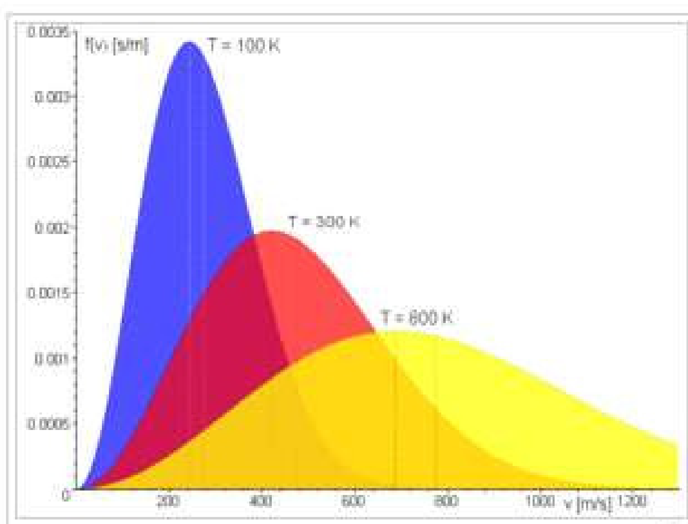

Fig. 7.1 *Probability Measure*

Definition

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The requirements for a function μ to be a probability measure on a probability space are that:

- μ must return results in the unit interval [0, 1] returning 0 for the empty set and 1 for the entire space.
- μ must satisfy the countable additively property that for all countable collections $\{E_i\}$ of pairwise disjoint sets:

$$\mu\Big(\bigcup_{i\in I} E_i\Big) = \sum_{i\in I} \mu(E_i).$$

For example, given three elements 1, 2 and 3 with probabilities 1/4, 1/4 and 1/2, the value assigned to {1, 3} is 1/4 + 1/2 = 3/4, as in the diagram on the right. The conditional probability based on the intersection of events defined as:

$$\mu(B \mid A) = \frac{\mu(A \cap B)}{\mu(A)}.$$

Satisfies the probability measure requirements so long as $\mu(A)$ is not zero.

Probability measures are distinct from the more general notion of fuzzy measures in which there is no requirement that the fuzzy values sum up to 1 and the additive property is replaced by an order relation based on set inclusion.

Example Applications

Market measures which assign probabilities to financial market spaces based on actual market movements are examples of probability measures which are of interest in mathematical finance, e.g., in the pricing of financial derivatives. For instance, a risk-neutral measure is a probability measure which assumes that the current value of assets is the expected value of the future payoff taken with respect to that same risk neutral measure (i.e., calculated using the corresponding risk neutral density function), and discounted at the risk-free rate. If there is a unique probability measure that must be used to price assets in a market, then the market is called a complete market.

Not all measures that intuitively represent chance or likelihood are probability measures. For instance, although the fundamental concept of a system in statistical mechanics is a measure space, such measures are not always probability measures. In general, in statistical physics, if we consider sentences of the form 'The Probability of a System S Assuming State A is P' the geometry of the system does not always lead to the definition of a probability measure under congruence, although it may do so in the case of systems with just one degree of freedom.

NOTES

Probability measures are also used in mathematical biology. For instance, in comparative sequence analysis a probability measure may be defined for the likelihood that a variant may be permissible for an amino acid in a sequence.

Ultrafilters can be understood as {0, 1}-valued probability measures, allowing for many intuitive proofs based upon measures. For instance, Hindman's Theorem can be proven from the further investigation of these measures, and their convolution in particular.

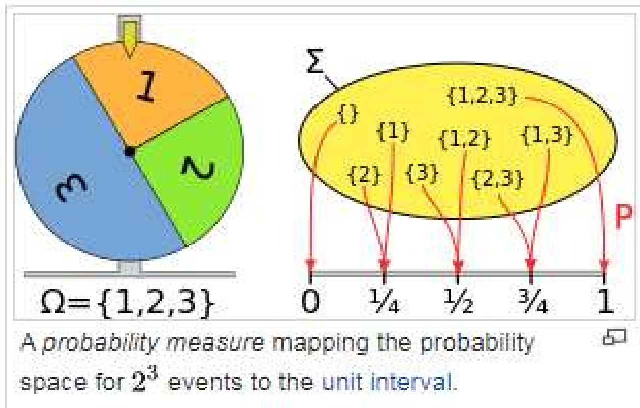

***Fig.** 7.2 Unit Interval*

Check Your Progress

1. Define the fuzzy measure.
2. What do you understand by the belief and plausibility measures?
3. Explain the probability theory.
4. Elaborate on the probability measure.

7.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Fuzzy measure theory considers generalized measures in which the additive property is replaced by the weaker property of monotonicity. The central concept of fuzzy measure theory is the fuzzy measure which was introduced by Choquet in 1953 and independently defined by Sugeno in 1974 in the context of fuzzy integrals. There exists a number of different classes of fuzzy measures including plausibility/belief measures; possibility/necessity measures; and probability measures which are a subset of classical measures.
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NOTES

Dempster–Shafer Theory (DST). They stated that there is a close relation between IFS theory and DST evidence theory, allowing them to aggregate local criteria without restriction. They also proposed a method for resolving these issues based on the interpretation of IFSs in the context of DST. We provide Belief (Bel) and Plausibility (Pl) measurements on IFSs in the context of DST in this research.

3. Probability theory is the branch of mathematics concerned with probability. Although there are several different probability interpretations, probability theory treats the concept in a rigorous mathematical manner by expressing it through a set of axioms.
4. A probability measure is a real-valued function defined on a set of events in a probability space that satisfies measure properties, such as countable additively. The difference between a probability measure and the more general notion of measure (which includes concepts like area or volume) is that a probability measure must assign value 1 to the entire probability space. Intuitively, the additively property says that the probability assigned to the union of two disjoint events by the measure should be the sum of the probabilities of the events.

7.6 SUMMARY

- Fuzzy measure theory considers generalized measures in which the additive property is replaced by the weaker property of monotonicity.
- The central concept of fuzzy measure theory is the fuzzy measure which was introduced by Choquet in 1953 and independently defined by Sugeno in 1974 in the context of fuzzy integrals.
- When a fuzzy measure is used to define a function, such as the Sugeno integral or Choquet integral, these properties will be crucial in understanding the function's behavior.
- A symmetric fuzzy measure will result in the Ordered Weighted Averaging (OWA) operator. Submodular fuzzy measures result in convex functions, while supermodular fuzzy measures result in concave functions when used to define a Choquet integral.
- A discrete fuzzy measure g on a set X is called k-additive $(1 \leq k \leq |X|)$ if its Möbius representation verifies M (E) = 0, whenever E > k for any E $\subseteq$ X and there exists a subset F with k elements such that M (F) $\neq$ 0.
- In game theory, the Shapley value or Shapley index is used to indicate the weight of a game.
- Shapley values can be calculated for fuzzy measures in order to give some indication of the importance of each singleton.

NOTES

- Probability has traditionally been used to model uncertainty in the presence of unpredictability.
- Dempster–Shafer Theory (DST)-based belief and plausibility metrics have arisen as another approach of quantifying uncertainty, and they have been intensively explored and utilized in a variety of fields.
- Hwang and Yang suggested a fuzzy integral-based generalization of belief and plausibility functions to IFSs, which captures additional information on the shift in intuitionistic fuzzy focal elements on DST.
- Hwang and Yang stated that there is a close relation between IFS theory and DST evidence theory, allowing them to aggregate local criteria without restriction. They also proposed a method for resolving these issues based on the interpretation of IFSs in the context of DST. We provide belief (Bel) and plausibility (Pl) measurements on IFSs in the context of DST in this research.
- Dempster first introduced the lower and higher probability via a multivalued mapping. Shafer built belief and plausibility metrics on ordinary subsets based on Dempster's lower and upper probability a decade later.
- Probability theory is the branch of mathematics concerned with probability. Although there are several different probability interpretations, probability theory treats the concept in a rigorous mathematical manner by expressing it through a set of axioms.
- Axioms formalise probability in terms of a probability space, which assigns a measure taking values between 0 and 1, termed the probability measure, to a set of outcomes called the sample space.
- As a mathematical foundation for statistics, probability theory is essential to many human activities that involve quantitative analysis of data.
- A probability measure is a real-valued function defined on a set of events in a probability space that satisfies measure properties, such as countable additively.
- The difference between a probability measure and the more general notion of measure (which includes concepts like area or volume) is that a probability measure must assign value 1 to the entire probability space.
- The additively property says that the probability assigned to the union of two disjoint events by the measure should be the sum of the probabilities of the events.

7.7 KEY WORDS

- **Fuzzy measure:** Fuzzy measure theory considers generalized measures in which the additive property is replaced by the weaker property of monotonicity. The central concept of fuzzy measure theory is the fuzzy measure which was introduced by Choquet in 1953 and independently defined by Sugeno in 1974 in the context of fuzzy integrals.

NOTES

- **Möbius representation:** Möbius representation can be used to give an indication of which subsets of X interact with one another. For instance, an additive fuzzy measure has Möbius values all equal to zero except for singletons. The fuzzy measure *g* in standard representation can be recovered from the Möbius form using the Zeta transform:

$$g(E) = \sum_{F \subseteq E} M(F), \forall E \subseteq \mathbf{X}.$$

- **Shapley values:** Shapley values can be calculated for fuzzy measures in order to give some indication of the importance of each singleton.
- **Probability theory:** Probability theory is the branch of mathematics concerned with probability. Although there are several different probability interpretations, probability theory treats the concept in a rigorous mathematical manner by expressing it through a set of axioms.
- **Sample space:** Axioms formalise probability in terms of a probability space, which assigns a measure taking values between 0 and 1, termed the probability measure, to a set of outcomes called the sample space.
- **Probability measure:** A probability measure is a real-valued function defined on a set of events in a probability space that satisfies measure properties, such as countable additively. The difference between a probability measure and the more general notion of measure is that a probability measure must assign value 1 to the entire probability space.

7.8 SELF-ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. What is fuzzy measure?
2. Define plausibility measures.
3. What do you mean by the probability measure?

Long-Answer Questions

1. Explain the fuzzy measure and its significance with the help of appropriate examples.

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2. Briefly discuss the basic concept of belief and plausibility measures with the help of appropriate examples.
3. What do you understand by the probability measure? Explain giving appropriate examples.

7.9 FURTHER READINGS

Klir, George J. and Bo Yuan. 1995. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. New Jersey: Prentice Hall Inc.

Klir, George J. and Tina A. Folger. 2009. *Fuzzy Sets, Uncertainty and Information*. New Delhi: Prentice Hall of India.

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UNIT 8 POSSIBILITY MEASURES AND NECESSITY MEASURES

NOTES

Structure

8.0 INTRODUCTION

Possibility theory is a mathematical theory for dealing with certain types of uncertainty and is an alternative to probability theory. It uses measures of possibility and necessity between 0 and 1, ranging from impossible to possible and unnecessary to necessary, respectively. Professor Lotfi Zadeh first introduced possibility theory in 1978 as an extension of his theory of fuzzy sets and fuzzy logic. Didier Dubois and Henri Prade further contributed to its development. Earlier in the 1950s, economist G. L. S. Shackle proposed the min/max algebra to describe degrees of potential surprise.

Possibility theory is an uncertainty theory dealing with the incomplete information. It is similar to probability theory because it is based on set-functions. It differs from the latter by the use of a pair of dual set functions instead of only one. The name "Theory of Possibility" was created by Zadeh, who was inspired by a paper by Gaines and Kohout.

Possibility theory is engaged with the principle of minimal specificity. It states that any hypothesis not known to be impossible cannot be ruled out. A possibility distribution À is said to be at least as specific as another À‘ if and only if for each state of affairs s: $\pi(s) \leq \pi$ (s). Then, π is at least as preventive and informative as π.

Possibility theory has permitted a typology of fuzzy rules to be laid bare, distinguishing rules whose persistence is to propagate uncertainty through reasoning steps, from rules whose main purpose is similarity-based interpolation, depending on the choice of a many-valued implication connective that models a rule.

In this unit, you will study about the possibility measures, necessity measures, and relationship among classes of fuzzy measures.

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8.1 OBJECTIVES

After going through this unit, you will be able to:

- Elaborate on the possibility measures
- Define the necessity measures
- Analyse the relationship among classes of fuzzy measures

8.2 POSSIBILITY MEASURES

For simplicity, assume that the universe of discourse Ω is a finite set. A possibility measure is a function pos from 2^{Ω} to [0, 1] such that:

Axiom 1: $\mathrm{pos}(\varnothing) = 0$

Axiom 2: $\mathrm{pos}(\Omega) = 1$

Axiom 3: $\mathrm{pos}(U \cup V) = \max\left(\mathrm{pos}(U), \mathrm{pos}(V)\right)$ for any disjoint subsets U and V.

It follows that, like probability, the possibility measure is determined by its behaviour on singletons:

$$\mathrm{pos}(U) = \max_{\omega \in U} \mathrm{pos}(\{\omega\})$$

Provided that U is finite or countable infinite.

Axiom 1 can be interpreted as the assumption that Ω is an exhaustive description of future states of the world, because it means that no belief weight is given to elements outside Ω.

Axiom 2 could be interpreted as the assumption that the evidence from which pos was constructed is free of any contradiction. Technically, it implies that there is at least one element in Ω with possibility 1.

Axiom 3 corresponds to the additively axiom in probabilities. However there is an important practical difference. Possibility theory is computationally more convenient because Axioms 1–3 imply that:

$$\mathrm{pos}(U \cup V) = \max\left(\mathrm{pos}(U), \mathrm{pos}(V)\right) \text{ for any subsets } U \text{ and } V.$$

Because one can know the possibility of the union from the possibility of each component, it can be said that possibility is compositional with respect to the union operator. Note however that it is not compositional with respect to the intersection operator. Generally:

$$\operatorname{pos}(U \cap V) \leq \min\left(\operatorname{pos}(U), \operatorname{pos}(V)\right) \leq \max\left(\operatorname{pos}(U), \operatorname{pos}(V)\right).$$

When Ω is not finite, Axiom 3 can be replaced by:

For all index sets I, if the subsets $U_{i,\, i \in I}$ are pairwise disjoint, $\operatorname{pos}(\cup_{i \in I} U_i) = \sup_{i \in I} \operatorname{pos}(U_i)$.

A possibility measure can be seen as a consonant plausibility measure in Dempster–Shafer theory of evidence. The operators of possibility theory can be seen as a hyper-cautious version of the operators of the transferable belief model, a modern development of the theory of evidence.

Possibility can be seen as an upper probability: any possibility distribution defines a unique credal set set of admissible probability distributions by

$$K = \{\, p : \forall S\ p(S) \leq \operatorname{pos}(S) \,\}.$$

This allows one to study possibility theory using the tools of imprecise probabilities.

For example, there is one bottle, either completely full or totally empty. The proposition "the possibility level that the bottle is full is 0.5" describes a degree of belief. One way to interpret 0.5 in that proposition is to define its meaning as: I am ready to bet that it's empty as long as the odds are even (1:1) or better, and I would not bet at any rate that it's full.

8.3 NECESSITY MEASURES

We call generalized necessity the dual of a generalized possibility. The generalized necessities are related with a very simple and interesting fuzzy logic we call necessity logic. In the deduction apparatus of necessity logic the logical axioms are the usual classical tautologies. Also, there is only a fuzzy inference rule extending the usual Modus Ponens. Such a rule says that if α and $\alpha \to \beta$ are proved at degree λ and μ, respectively, then we can assert β at degree min $\{\lambda,\mu\}$. It is easy to see that the theories of such a logic are the generalized necessities and that the completely consistent theories coincide with the necessities.

Whereas probability theory uses a single number, the probability, to describe how likely an event is to occur, possibility theory uses two concepts, the possibility and the necessity of the event. For any set U, the necessity measure is defined by

$$\operatorname{nec}(U) = 1 - \operatorname{pos}(\overline{U})$$

In the above formula, $\overline{U}$ denotes the complement of U, that is the elements of Ω that do not belong to U. It is straightforward to show that:

$$\text{nec}(U) \le \text{pos}(U) \text{ for any } U$$

And that:

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$$\text{nec}(U \cap V) = \min(\text{nec}(U), \text{nec}(V))$$

Note that contrary to probability theory, possibility is not self-dual. That is, for any event U, we only have the inequality:

$$\text{pos}(U) + \text{pos}(\overline{U}) \ge 1$$

However, the following duality rule holds:

For any event U, either $\text{pos}(U) = 1$ or $\text{nec}(U) = 0$

Accordingly, beliefs about an event can be represented by a number and a bit.

8.4 RELATIONSHIP AMONG THE CLASSES OF FUZZY MEASURES

There are four cases relationship among the classes of fuzzy measures that can be interpreted as follows:

1. $\text{nec}(U) = 1$ means that U is necessary. U is certainly true. It implies that $\text{pos}(U) = 1$.
2. $\text{pos}(U) = 0$ means that U is impossible. U is certainly false. It implies that $\text{nec}(U) = 0$.
3. $\text{pos}(U) = 1$ means that U is possible. I would not be surprised at all if U occurs. It leaves $\text{nec}(U)$ unconstrained.
4. $\text{nec}(U) = 0$ means that U is unnecessary. I would not be surprised at all if U does not occur. It leaves $\text{pos}(U)$ unconstrained.

The intersection of the last two cases is $\text{nec}(U) = 0$ and $\text{pos}(U) = 1$ meaning that I believe nothing at all about U. Because it allows for indeterminacy like this, possibility theory relates to the graduation of a many-valued logic, such as intuitionistic logic, rather than the classical two-valued logic.

Note that unlike possibility, fuzzy logic is compositional with respect to both the union and the intersection operator. The relationship with fuzzy theory can be explained with the following classical example.

Fuzzy logic: When a bottle is half full, it can be said that the level of truth of the proposition "The Bottle is Full" is 0.5. The word "Full" is seen as a fuzzy predicate describing the amount of liquid in the bottle.

NOTES

Check Your Progress

1. What do you understand by the possibility theory?
2. Explain the possibility measures.
3. Define the necessity measures.
4. Illustrate the relationship among the classes of fuzzy measures.

8.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Possibility theory is a mathematical theory for dealing with certain types of uncertainty and is an alternative to probability theory. It uses measures of possibility and necessity between 0 and 1, ranging from impossible to possible and unnecessary to necessary, respectively.
2. For simplicity, assume that the universe of discourse Ω is a finite set. A possibility measure is a function pos from 2^{Ω} to [0, 1] such that:

   Axiom 1: $\mathrm{pos}(\varnothing) = 0$

   Axiom 2: $\mathrm{pos}(\Omega) = 1$

   Axiom 3: $\mathrm{pos}(U \cup V) = \max(\mathrm{pos}(U), \mathrm{pos}(V))$ for any disjoint subsets U and V.
3. The necessity measure is defined by

   $$\mathrm{nec}(U) = 1 - \mathrm{pos}(\overline{U})$$

   In the above formula, $\overline{U}$ denotes the complement of U, that is the elements of Ω that do not belong to U.
4. The intersection of the last two cases is $\mathrm{nec}(U) = 0$ and $\mathrm{pos}(U) = 1$ meaning that I believe nothing at all about U. Because it allows for indeterminacy like this, possibility theory relates to the graduation of a many-valued logic, such as intuitionistic logic, rather than the classical two-valued logic.

8.6 SUMMARY

- Possibility theory is an uncertainty theory dealing with the incomplete information. It is similar to probability theory because it is based on set-functions.

NOTES

- For simplicity, assume that the universe of discourse Ω is a finite set. A possibility measure is a function pos from 2^{Ω} to [0, 1] such that:

  Axiom 1: $\operatorname{pos}(\varnothing) = 0$

  Axiom 2: $\operatorname{pos}(\Omega) = 1$

  Axiom 3: $\operatorname{pos}(U \cup V) = \max\left(\operatorname{pos}(U), \operatorname{pos}(V)\right)$ for any disjoint subsets U and V.
- A possibility measure can be seen as a consonant plausibility measure in Dempster–Shafer theory of evidence. The operators of possibility theory can be seen as a hyper-cautious version of the operators of the transferable belief model, a modern development of the theory of evidence.
- Possibility can be seen as an upper probability: any possibility distribution defines a unique credal set set of admissible probability distributions by

$$K = \{\, p : \forall S\; p(S) \leq \operatorname{pos}(S) \,\}\,.$$

- We call generalized necessity the dual of a generalized possibility. The generalized necessities are related with a very simple and interesting fuzzy logic we call necessity logic.
- The intersection of the last two cases is $\operatorname{nec}(U) = 0$ and $\operatorname{pos}(U) = 1$ meaning that I believe nothing at all about U. Because it allows for indeterminacy like this, possibility theory relates to the graduation of a many-valued logic, such as intuitionistic logic, rather than the classical two-valued logic.
- Note that unlike possibility, fuzzy logic is compositional with respect to both the union and the intersection operator. The relationship with fuzzy theory can be explained with the following classical example.

8.7 KEY WORDS

- **Possibility theory:** Possibility theory is a mathematical theory for dealing with certain types of uncertainty and is an alternative to probability theory.
- **Possibility measure:** A possibility measure can be seen as a consonant plausibility measure in Dempster–Shafer theory of evidence.
- **Necessity measure:** The necessity measure is defined by

$$\operatorname{nec}(U) = 1 - \operatorname{pos}(\overline{U})$$

  In the above formula, $\overline{U}$ denotes the complement of U, that is the elements of Ω that do not belong to U.

8.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

NOTES

Short-Answer Questions

1. Elaborate on the possibility theory.
2. Interpret the possibility measures.
3. Explain the necessity measures.
4. Define the relationship among the classes of fuzzy measures.

Long-Answer Questions

1. Briefly discuss the possibility measures.
2. Describe the necessity measures. Give appropriate examples.
3. Analyse the relationship among the classes of fuzzy measures.

8.9 FURTHER READINGS

Klir, George J. and Bo Yuan. 1995. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. New Jersey: Prentice Hall Inc.

Klir, George J. and Tina A. Folger. 2009. *Fuzzy Sets, Uncertainty and Information*. New Delhi: Prentice Hall of India.

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BLOCK - III
UNCERTAINTY AND MEASURES OF DISSONANCE

NOTES

UNIT 9 TYPES OF UNCERTAINTY

Structure

9.0 INTRODUCTION

Uncertainty refers to epistemic situations involving imperfect or unknown information. It applies to predictions of future events, to physical measurements that are already made, or to the unknown. Uncertainty arises in partially observable or stochastic environments, as well as due to ignorance, indolence, or both. It arises in any number of fields, including insurance, philosophy, physics, statistics, economics, finance, psychology, sociology, engineering, metrology, meteorology, ecology and information science.

Although, the terms are used in various ways among the general public, many specialists in decision theory, statistics and other quantitative fields have defined uncertainty. The lack of certainty, a state of limited knowledge where it is impossible to exactly describe the existing state, a future outcome, or more than one possible outcome. A set of possible states or outcomes where probabilities are assigned to each possible state or outcome – this also includes the application of a probability density function to continuous variables.

In statistics and economics, second-order uncertainty is represented in probability density functions over (first-order) probabilities. In economics, in 1921 Frank Knight distinguished uncertainty from risk with uncertainty being lack of knowledge which is immeasurable and impossible to calculate; this is now referred to as Knightian uncertainty:

Uncertainty must be taken in a sense radically distinct from the familiar notion of risk, from which it has never been properly separated.... The essential

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fact is that 'Risk' means in some cases a quantity susceptible of measurement, while at other times it is something distinctly not of this character; and there are far-reaching and crucial differences in the bearings of the phenomena depending on which of the two is really present and operating.... It will appear that a measurable uncertainty, or 'Risk' proper, as we shall use the term, is so far different from an unmeasurable one that it is not in effect an uncertainty at all.

— Frank Knight (1885–1972), Risk, Uncertainty, and Profit (1921), University of Chicago.

In this unit, you will study about the types of uncertainty, measures of fuzziness, and related problems.

9.1 OBJECTIVES

After going through this unit, you will be able to:

- Explain the types of uncertainty
- Understand the measures of fuzziness
- Solve the related problems

9.2 TYPES OF UNCERTAINTY

In scientific modelling, in which the prediction of future events should be understood to have a range of expected values. In physics, the Heisenberg uncertainty principle forms the basis of modern quantum mechanics. In engineering, uncertainty can be used in the context of validation and verification of material modeling. Uncertainty has been a common theme in art, both as a thematic device (see, for example, the indecision of Hamlet), and as a quandary for the artist (such as Martin Creed's difficulty with deciding what artworks to make).

Uncertainty in science, and science in general, may be interpreted differently in the public sphere than in the scientific community. This is due in part to the diversity of the public audience, and the tendency for scientists to misunderstand lay audiences and therefore not communicate ideas clearly and effectively. One example is explained by the information deficit model. Also, in the public realm, there are often many scientific voices giving input on a single topic. For example, depending on how an issue is reported in the public sphere, discrepancies between outcomes of multiple scientific studies due to methodological differences could be interpreted by the public as a lack of consensus in a situation where a consensus does in fact exist. This interpretation may have even been intentionally promoted, as scientific uncertainty may be managed to reach certain goals. For example, climate change deniers took the advice of Frank Luntz to frame global warming as an issue of scientific uncertainty, which was a precursor to the conflict frame used by journalists when reporting the issue.

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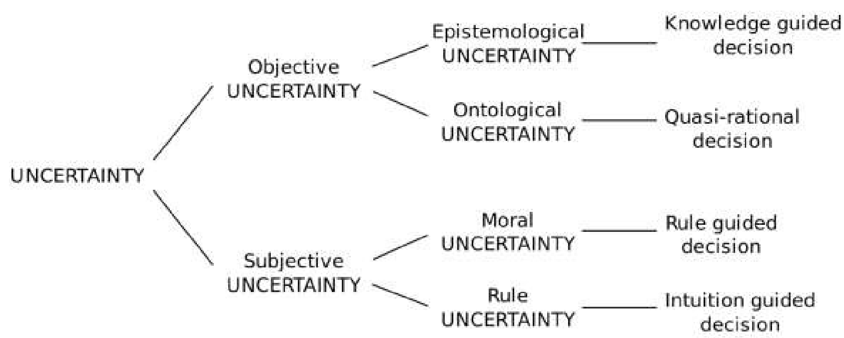
Uncertainty of a measurement can be determined by repeating a measurement to arrive at an estimate of the standard deviation of the values. Then, any single value has an uncertainty equal to the standard deviation. However, if the values are averaged, then the mean measurement value has a much smaller uncertainty, equal to the standard error of the mean, which is the standard deviation divided by the square root of the number of measurements. This procedure neglects systematic errors, however.



9.3 MEASURES OF FUZZINESS

In mathematics, fuzzy measure theory considers generalized measures in which the additive property is replaced by the weaker property of monotonicity. The central concept of fuzzy measure theory is the fuzzy measure which was introduced by Choquet in 1953 and independently defined by Sugeno in 1974 in the context of fuzzy integrals. There exists a number of different classes of fuzzy measures including plausibility/belief measures; possibility/necessity measures; and probability measures which are a subset of classical measures.

Let $\mathbf{X}$ be a universe of discourse, $\mathcal{C}$ be a class of subsets of $\mathbf{X}$ and $E, F \in \mathcal{C}$. A function $g : \mathcal{C} \to \mathbb{R}$ where

1. $\emptyset \in \mathcal{C} \Rightarrow g(\emptyset) = 0$
2. $E \subseteq F \Rightarrow g(E) \leq g(F)$

is called a fuzzy measure. A fuzzy measure is called normalized or regular if $g(\mathbf{X}) = 1$.

9.3.1 Properties of Fuzzy Measures

Understanding the properties of fuzzy measures is useful in application. When a fuzzy measure is used to define a function such as the Sugeno integral or Choquet integral, these properties will be crucial in understanding the function's behaviour. For instance, the Choquet integral with respect to an additive fuzzy measure reduces to the Lebesgue integral. In discrete cases, a symmetric fuzzy measure will result in the Ordered Weighted Averaging (OWA) operator. Submodular fuzzy measures result in convex functions, while supermodular fuzzy measures result in concave functions when used to define a Choquet integral.

A fuzzy measure is:

- Additive if for any $E, F \in \mathcal{C}$ such that $E \cap F = \emptyset$ we have $g(E \cup F) = g(E) + g(F)$.
- Supermodular if for any $E, F \in \mathcal{C}$, we have $g(E \cup F) + g(E \cap F) \geq g(E) + g(F)$;
- Submodular if for any $E, F \in \mathcal{C}$, we have $g(E \cup F) + g(E \cap F) \leq g(E) + g(F)$;
- Superadditive if for any $E, F \in \mathcal{C}$, such that $E \cap F = \emptyset$, we have $g(E \cup F) \geq g(E) + g(F)$;
- Subadditive if for any $E, F \in \mathcal{C}$, such that $E \cap F = \emptyset$, we have $g(E \cup F) \leq g(E) + g(F)$;
- Symmetric if for any $E, F \in \mathcal{C}$, we have $|E| = |F|$ implies $g(E) = g(F)$;
- Boolean if for any $E \in \mathcal{C}$, we have $g(E) = 0$ or $g(E) = 1$.

Fuzzy measures are defined on a semi ring of sets or monotone class which may be as granular as the power set of X, and even in discrete cases the number of variables can as large as $2^{|X|}$. For this reason, in the context of multi-criteria decision analysis and other disciplines, simplification assumptions on the fuzzy measure have been introduced so that it is less computationally expensive to determine and use. For instance, when it is assumed the fuzzy measure is additive, it will hold that

$$g(E) = \sum_{i \in E} g(\{i\})$$

And the values of the fuzzy measure can be evaluated from the values on X. Similarly, a symmetric fuzzy measure is defined uniquely by |X| values. Two important fuzzy measures that can be used are the Sugeno- or λ-fuzzy measure and k-additive measures, introduced by Sugeno and Grabisch, respectively.

9.3.2 Möbius Representation of Fuzzy Measure

Let *g* be a fuzzy measure, the Möbius representation of *g* is given by the set function *M*, where for every $E, F \subseteq X$.

$$M(E) = \sum_{F \subseteq E} (-1)^{|E \setminus F|} g(F).$$

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The equivalent axioms in Möbius representation are:

1. $M(\emptyset) = 0$.
2. $\sum_{F \subseteq E | i \in F} M(F) \geq 0$, for all $E \subseteq \mathbf{X}$ and all $i \in E$

A fuzzy measure in Möbius representation M is called normalized if

$$\sum_{E \subseteq \mathbf{X}} M(E) = 1.$$

Möbius representation can be used to give an indication of which subsets of X interact with one another. For instance, an additive fuzzy measure has Möbius values all equal to zero except for singletons. The fuzzy measure g in standard representation can be recovered from the Möbius form using the Zeta transform:

$$g(E) = \sum_{F \subseteq E} M(F), \forall E \subseteq \mathbf{X}.$$

Check Your Progress

1. Define the different types of uncertainty.
2. Explain the measures of fuzziness.
3. Illustrate the properties of fuzzy measures.
4. Elaborate on the Möbius representation of fuzzy measure.

9.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. In scientific modelling, in which the prediction of future events should be understood to have a range of expected values. In physics, the Heisenberg uncertainty principle forms the basis of modern quantum mechanics. In engineering, uncertainty can be used in the context of validation and verification of material modeling.
2. Fuzzy measure theory considers generalized measures in which the additive property is replaced by the weaker property of monotonicity. The central concept of fuzzy measure theory is the fuzzy measure which was introduced

by Choquet in 1953 and independently defined by Sugeno in 1974 in the context of fuzzy integrals.

3. Understanding the properties of fuzzy measures is useful in application. When a fuzzy measure is used to define a function such as the Sugeno integral or Choquet integral, these properties will be crucial in understanding the function's behaviour.
4. Let g be a fuzzy measure, the Möbius representation of g is given by the set function M, where for every $E, F \subseteq X$.

$$M(E) = \sum_{F \subseteq E} (-1)^{|E \setminus F|} g(F).$$

9.5 SUMMARY

- In scientific modelling, in which the prediction of future events should be understood to have a range of expected values. In physics, the Heisenberg uncertainty principle forms the basis of modern quantum mechanics.
- In engineering, uncertainty can be used in the context of validation and verification of material modeling.
- Uncertainty in science, and science in general, may be interpreted differently in the public sphere than in the scientific community. This is due in part to the diversity of the public audience, and the tendency for scientists to misunderstand lay audiences and therefore not communicate ideas clearly and effectively.
- Uncertainty of a measurement can be determined by repeating a measurement to arrive at an estimate of the standard deviation of the values. Then, any single value has an uncertainty equal to the standard deviation.
- Fuzzy measure theory considers generalized measures in which the additive property is replaced by the weaker property of monotonicity. The central concept of fuzzy measure theory is the fuzzy measure which was introduced by Choquet in 1953 and independently defined by Sugeno in 1974 in the context of fuzzy integrals.
- There exists a number of different classes of fuzzy measures including plausibility/belief measures; possibility/necessity measures; and probability measures which are a subset of classical measures.
- Understanding the properties of fuzzy measures is useful in application. When a fuzzy measure is used to define a function such as the Sugeno integral or Choquet integral, these properties will be crucial in understanding the function's behaviour.

NOTES

- A symmetric fuzzy measure will result in the Ordered Weighted Averaging (OWA) operator. Submodular fuzzy measures result in convex functions, while supermodular fuzzy measures result in concave functions when used to define a Choquet integral.
- Fuzzy measures are defined on a semi ring of sets or monotone class which may be as granular as the power set of X, and even in discrete cases the number of variables can as large as $2^{|X|}$.
- Möbius representation can be used to give an indication of which subsets of *X* interact with one another. For instance, an additive fuzzy measure has Möbius values all equal to zero except for singletons.

9.6 KEY WORDS

- **Types of uncertainty:** In scientific modelling, in which the prediction of future events should be understood to have a range of expected values. In physics, the Heisenberg uncertainty principle forms the basis of modern quantum mechanics.
- **Measures of fuzziness:** Fuzzy measures are defined on a semi ring of sets or monotone class which may be as granular as the power set of X, and even in discrete cases the number of variables can as large as $2^{|X|}$.
- **Properties of fuzzy measures:** Understanding the properties of fuzzy measures is useful in application. When a fuzzy measure is used to define a function such as the Sugeno integral or Choquet integral, these properties will be crucial in understanding the function's behaviour.
- **Möbius representation:** Möbius representation can be used to give an indication of which subsets of *X* interact with one another. For instance, an additive fuzzy measure has Möbius values all equal to zero except for singletons.

9.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Interpret the different types of uncertainty.
2. Define the measures of fuzziness.
3. State the properties of fuzzy measures.
4. Explain the Möbius representation of fuzzy measure.

Long-Answer Questions

1. Briefly define the types of uncertainty with the help of examples.
2. Describe the measures of fuzziness. Give appropriate example.
3. Explain the properties of fuzzy measures.
4. Analyse the Möbius representation of fuzzy measure.

NOTES

9.8 FURTHER READINGS

Klir, George J. and Bo Yuan. 1995. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. New Jersey: Prentice Hall Inc.

Klir, George J. and Tina A. Folger. 2009. *Fuzzy Sets, Uncertainty and Information*. New Delhi: Prentice Hall of India.

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UNIT 10 CLASSICAL MEASURES OF UNCERTAINTY

NOTES

Structure

10.0 INTRODUCTION

This is a set of possible states or outcomes where probabilities are assigned to each possible state or outcome – this also includes the application of a probability density function to continuous variables. Measurement uncertainty is a quantitative sign of the quality of measurement outcomes, without that, they could not be compared between themselves, with specified reference values or to a standard. Uncertainty assessment is important to assurance the metrological traceability of measurement results and to ensure that they are accurate and reliable.

In 1928, Hartley presented a simple measure of information. The Hartley function is a measure of uncertainty, introduced by Ralph Hartley in 1928. If a sample from a finite set A uniformly at random is picked, the information revealed after the outcome is known is given by the Hartley function. The Hartley (symbol Hart), also called a ban, or a dit (short for decimal digit), is a logarithmic unit that measures information or entropy, based on base 10 logarithms and powers of 10. One Hartley is the information content of an event if the probability of that event occurring is 1D 10. It is therefore equal to the information contained in one decimal digit (or dit), assuming a priori equiprobability of each possible value. It is named after Ralph Hartley.

In information theory, the entropy of a random variable is the average level of "Information", "Surprise", or "Uncertainty" inherent in the variable's possible outcomes. The concept of information entropy was introduced by Claude Shannon in his 1948 paper "A Mathematical Theory of Communication", and is sometimes called Shannon entropy in his honour.

Ludwig Boltzmann defined entropy as a measure of the number of possible microscopic states (microstates) of a system in thermodynamic equilibrium, consistent with its macroscopic thermodynamic properties, which constitute the macrostate of the system.

In this unit, you will study about the classical measures of uncertainty, the Hartley information, the Shannon entropy, and the Boltzmann entropy.

10.1 OBJECTIVES

After going through this unit, you will be able to:

- Define the classical measures of uncertainty
- Elaborate on the Hartley information
- Understand the Shannon entropy
- Explain the Boltzmann entropy

10.2 CLASSICAL MEASURES OF UNCERTAINTY

Fuzzy Evidence Theory (FET), or Fuzzy Dempster-Shafer Theory (FDST) states the three types of **uncertainty**, i.e., fuzziness, non-specificity and conflict, which are typically limited within one framework. Consequently, it is acknowledged as the utmost favourable methods for practical applications.

Dempster-Shafer theory (DST) or Evidence Theory (ET) is accepted as a flexible framework to model various processes of quantitative reasoning and decision-making under uncertainty. It is extensively used in practical applications, such as function approximation, regression analysis, risk analysis, etc., where DST framework is capable of handling different types of non-specificity and also conflict while modeling under uncertainty.

Computing a Fuzzy Body of Evidence (FBoE) can quantify different sort of uncertainties, such as imprecision, discord and degree of confidence.

Uncertainty Measure in the Fuzzy Evidence Framework

For the fuzzy evidence framework, two main measures have been recommended. The first measure is the General Uncertainty Measure (GM), which was introduced by Liu.

The General Uncertainty Measure (GM) for Fuzzy Body of Evidence (FBoE) is defined as,

$$GM\left(FBoE\right) \equiv$$

$$-\sum_{x\in\Theta}\left[BetP\left(x\right)log_2 BetP(x) + \overline{BetP}\left(x\right)log_2\overline{BetP}\left(x\right)\right] \qquad (10.1)$$

Where,

$$BetP(x) \equiv \sum_{i=1}^{f} \frac{m(\widetilde{A}_i)\mu_{\widetilde{A}_i}(x)}{\sum_{x' \in S_{\widetilde{A}_i}} \mu_{\widetilde{A}_i}(x')}, \overline{BetP}(x) \equiv \sum_{i=1}^{f} \frac{m(\widetilde{A}_i)(1-\mu_{\widetilde{A}_i}(x))}{\sum_{x' \in S_{\widetilde{A}_i}} \mu_{\widetilde{A}_i}(x')}.$$

The second one is the Hybrid Entropy (FH), recommended by Zhu and Basir. It is defined as a measure which quantifies the overall uncertainty contained in a fuzzy evidence structure, for which the equation of the form is,

$$FH\left(FBoE\right) \equiv -\sum_{i=1}^{f} m(\widetilde{A}_i) log_2 \left(m(\widetilde{A}_i)(1-F(\widetilde{A}_i))\right) \qquad (10.2)$$

Where $F\left(\widetilde{A}\right)$ denotes the fuzzy entropy of fuzzy set $\widetilde{A}$ as:

$$F\left(\widetilde{A}\right) \equiv \frac{1}{|S_{\widetilde{A}}|} \sum_{x \in S_{\widetilde{A}}} \frac{\mu_{\widetilde{A} \cap \overline{\widetilde{A}}}(x)}{\mu_{\widetilde{A} \cup \overline{\widetilde{A}}}(x)}$$

The smaller is $F\left(\widetilde{A}\right)$, the less fuzzy is the fuzzy set $\widetilde{A}$. In Equation (10.1) and Equation (10.2), $S\left(\widetilde{A}_j\right)$ specifies the support of a fuzzy set $\widetilde{A}_j$ which means a crisp set that contains all such points $x \in \Theta$ for which $\mu_{\widetilde{A}_j}\left(x\right) > 0.$.

The above defined two measures are considered as the extension of the ambiguity measures, i.e., non-specificity and conflict, to the fuzzy set. Consequently, they can only solve the types of non-specificity and conflict in fuzzy evidence theory.

10.3 HARTLEY INFORMATION

In 1928, Hartley presented a simple measure of information. The term Hartley is named after Ralph Hartley, who suggested in 1928 to measure information using a logarithmic base equal to the number of distinguishable states in its representation, which would be the base 10 for a decimal digit. The Hartley (symbol Hart), also called a ban, or a dit (short for decimal digit), is a logarithmic unit that measures information or entropy, based on base 10 logarithms and powers of 10. One Hartley is the information content of an event if the probability of that event occurring is 1D 10. It is therefore equal to the information contained in one decimal digit (or dit), assuming a priori equiprobability of each possible value.

The Hartley function is a measure of uncertainty, introduced by Ralph Hartley in 1928. If a sample from a finite set A uniformly at random is picked, the information revealed after the outcome is known is given by the Hartley function

$$H_0(A) := \log_b |A|,$$

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Where $|A|$ denotes the cardinality of A.

If the base of the logarithm is 2, then the unit of uncertainty is the Shannon (more commonly known as bit). If it is the natural logarithm, then the unit is the nat. Hartley used a base-ten logarithm, and with this base, the unit of information is called the Hartley (aka ban or dit) in his honour. It is also known as the Hartley entropy.

We want to show that the Hartley function, $\log_2(n)$, is the only function mapping natural numbers to real numbers that satisfies

1. $H(mn) = H(m) + H(n)$ (additivity)
2. $H(m) \leq H(m+1)$ (monotonicity)
3. $H(2) = 1$ (normalization)

Let f be a function on positive integers that satisfies the above three properties. From the additive property, we can show that for any integer n and k,

$$f(n^k) = kf(n)$$

Let a, b, and t be any positive integers. There is a unique integer s determined by

$$a^s \leq b^t \leq a^{s+1} \qquad \ldots 10.3$$

Therefore,

$$s \log_2 a \leq t \log_2 b \leq (s+1) \log_2 a$$

And,

$$\frac{s}{t} \leq \frac{\log_2 b}{\log_2 a} \leq \frac{s+1}{t}$$

On the other hand, by monotonicity,

$$f(a^s) \leq f(b^t) \leq f(a^{s+1})$$

Using Equation (10.3), one gets

$$sf(a) \leq tf(b) \leq (s+1)f(a)$$

And,

$$\frac{s}{t} \leq \frac{f(b)}{f(a)} \leq \frac{s+1}{t}.$$

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Hence,

$$\left|\frac{f(b)}{f(a)} - \frac{\log_2(b)}{\log_2(a)}\right| \leq \frac{1}{t}$$

Since t can be arbitrarily large, the difference on the left hand side of the above inequality must be zero,

$$\frac{f(b)}{f(a)} = \frac{\log_2(b)}{\log_2(a)}$$

So,

$$f(a) = \mu \log_2(a)$$

For some constant ¼, which must be equal to 1 by the normalization property.

10.4 SHANNON ENTROPY

The concept of information entropy was introduced by Claude Shannon in his 1948 paper "A Mathematical Theory of Communication", and is sometimes called Shannon entropy in his honour. As an example, consider a biased coin with probability p of landing on heads and probability $1 - p$ of landing on tails. The maximum surprise is for $p = 1/2$, when there is no reason to expect one outcome over another, and in this case a coin flip has an entropy of one bit. The minimum surprise is when $p = 0$ or $p = 1$, when the event is known and the entropy is zero bits. Other values of p give different entropies between zero and one bits. The entropy of a random variable is the average level of "Information", "Surprise", or "Uncertainty" inherent in the variable's possible outcomes.

Named after Boltzmann's —-theorem, Shannon defined the entropy — (Greek capital letter eta) of a discrete random variable X with possible values $\{x_1, \ldots, x_n\}$ and probability mass function $\mathrm{P}(X)$ as:

$$\mathrm{H}(X) = \mathrm{E}[\mathrm{I}(X)] = \mathrm{E}[-\log(\mathrm{P}(X))].$$

Here E is the expected value operator, and I is the information content of X. I(X) is itself a random variable.

The entropy can explicitly be written as:

$$\mathrm{H}(X) = -\sum_{i=1}^{n} \mathrm{P}(x_i) \log_b \mathrm{P}(x_i)$$

Where b is the base of the logarithm used. Common values of b are 2, Euler's number e, and 10, and the corresponding units of entropy are the bits for $b = 2$, nats for $b = e$, and bans for $b = 10$.

In the case of $P(x_i) = 0$ for some i, the value of the corresponding summand $0 \log_b(0)$ is taken to be 0, which is consistent with the limit:

$$\lim_{p \to 0^+} p \log(p) = 0.$$

One may also define the conditional entropy of two variables X and Y taking values x_i and y_j, respectively, as:

$$\mathrm{H}(X|Y) = -\sum_{i,j} p(x_i, y_j) \log \frac{p(x_i, y_j)}{p(y_j)}$$

Where $p(x_i, y_j)$ is the probability that $X = x_i$ and $Y = y_j$. This quantity should be understood as the amount of randomness in the random variable X given the random variable Y.

The entropy was originally created by Shannon as part of his theory of communication, in which a data communication system is composed of three elements: a source of data, a communication channel, and a receiver. In Shannon's theory, the "Fundamental Problem of Communication" – as expressed by Shannon – is for the receiver to be able to identify what data was generated by the source, based on the signal it receives through the channel. Shannon considered various ways to encode, compress, and transmit messages from a data source, and proved in his famous source coding theorem that the entropy represents an absolute mathematical limit on how well data from the source can be losslessly compressed onto a perfectly noiseless channel. Shannon strengthened this result considerably for noisy channels in his noisy-channel coding theorem.

Shannon's definition of entropy, when applied to an information source, can determine the minimum channel capacity required to reliably transmit the source as encoded binary digits. Shannon's entropy measures the information contained in a message as opposed to the portion of the message that is determined (or predictable). Examples of the latter include redundancy in language structure or statistical properties relating to the occurrence frequencies of letter or word pairs, triplets etc.

10.5 BOLTZMANN ENTROPY

The concept entropy was first developed by German physicist Rudolf Clausius in the mid-nineteenth century as a thermodynamic property that predicts that certain spontaneous processes are irreversible or impossible. In statistical mechanics,

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entropy is formulated as a statistical property using probability theory. The statistical entropy perspective was introduced in 1870 by Austrian physicist Ludwig Boltzmann, who established a new field of physics that provided the descriptive linkage between the macroscopic observation of nature and the microscopic view based on the rigorous treatment of a large ensembles of microstates that constitute thermodynamic systems.

Ludwig Boltzmann defined entropy as a measure of the number of possible microscopic states (microstates) of a system in thermodynamic equilibrium, consistent with its macroscopic thermodynamic properties, which constitute the macrostate of the system. The easily measurable parameters volume, pressure, and temperature of the gas describe its macroscopic condition (state). At a microscopic level, the gas consists of a vast number of freely moving atoms or molecules, which randomly collide with one another and with the walls of the container. The collisions with the walls produce the macroscopic pressure of the gas, which illustrates the connection between microscopic and macroscopic phenomena.

The large number of particles of the gas provides an infinite number of possible microstates for the sample, but collectively they exhibit a well-defined average of configuration, which is exhibited as the macrostate of the system, to which each individual microstate contribution is negligibly small. The ensemble of microstates comprises a statistical distribution of probability for each microstate, and the maximum, a group of most probable configurations provide the macroscopic state.

Therefore, the system can be described as a whole by only a few macroscopic parameters, called the thermodynamic variables: the total energy E, volume V, pressure P, temperature T, and so forth. However, this description is only relatively simple when the system is in a state of equilibrium. Equilibrium may be illustrated with a simple example of a drop of food colouring falling into a glass of water. The dye diffuses in a complicated manner, which is difficult to precisely predict. However, after sufficient time has passed, the system reaches a uniform colour, a state much easier to explain.

Boltzmann formulated a simple relationship between entropy and the number of possible microstates of a system, which is denoted by the symbol Ω. The entropy S is proportional to the natural logarithm of this number.

$$S = k_{\mathrm{B}} \ln \Omega \cdot$$

The proportionality constant k_B is one of the fundamental constants of physics, and is named in honour of its discoverers, the Boltzmann constant.

Since Ω is a natural number (1,2,3,...), entropy is either zero or positive ($\ln(1) = 0$, $\ln \Omega \geq 0$). Boltzmann's entropy describes the system when all the accessible microstates are equally likely. It is the configuration corresponding to the maximum of entropy at equilibrium. The randomness or disorder is maximal, and so is the lack of distinction (or information) of each microstate.

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Entropy is a thermodynamic property just the same as pressure, volume, or temperature. Therefore, it connects the microscopic and the macroscopic world view. Boltzmann's principle is regarded as the foundation of statistical mechanics.

Check Your Progress

1. Explain the classical measures of uncertainty.
2. Illustrate the Hartley information.
3. Define the Shannon entropy.
4. Elaborate on the Boltzmann entropy.

10.6 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Fuzzy Evidence Theory (FET), or Fuzzy Dempster-Shafer Theory (FDST) states the three types of uncertainty, i.e., fuzziness, non-specificity and conflict, which are typically limited within one framework. Consequently, it is acknowledged as the utmost favourable methods for practical applications.
2. In 1928, Hartley presented a simple measure of information. The term Hartley is named after Ralph Hartley, who suggested in 1928 to measure information using a logarithmic base equal to the number of distinguishable states in its representation, which would be the base 10 for a decimal digit.
3. The concept of information entropy was introduced by Claude Shannon in his 1948 paper "A Mathematical Theory of Communication", and is sometimes called Shannon entropy in his honour.
4. Ludwig Boltzmann defined entropy as a measure of the number of possible microscopic states (microstates) of a system in thermodynamic equilibrium, consistent with its macroscopic thermodynamic properties, which constitute the macrostate of the system.

10.7 SUMMARY

- Fuzzy Evidence Theory (FET), or Fuzzy Dempster-Shafer Theory (FDST) states the three types of uncertainty, i.e., fuzziness, non-specificity and conflict, which are typically limited within one framework. Consequently, it is acknowledged as the utmost favourable methods for practical applications.

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- In 1928, Hartley presented a simple measure of information. The term Hartley is named after Ralph Hartley, who suggested in 1928 to measure information using a logarithmic base equal to the number of distinguishable states in its representation, which would be the base 10 for a decimal digit.
- The Hartley (symbol Hart), also called a ban, or a dit (short for decimal digit), is a logarithmic unit that measures information or entropy, based on base 10 logarithms and powers of 10. One Hartley is the information content of an event if the probability of that event occurring is 1/10.
- The Hartley function is a measure of uncertainty, introduced by Ralph Hartley in 1928. If a sample from a finite set A uniformly at random is picked, the information revealed after the outcome is known is given by the Hartley function

$$H_0(A) := \log_b |A|,$$

Where $|A|$ denotes the cardinality of A.

- The concept of information entropy was introduced by Claude Shannon in his 1948 paper "A Mathematical Theory of Communication", and is sometimes called Shannon entropy in his honour.
- The entropy was originally created by Shannon as part of his theory of communication, in which a data communication system is composed of three elements: a source of data, a communication channel, and a receiver.
- Shannon's definition of entropy, when applied to an information source, can determine the minimum channel capacity required to reliably transmit the source as encoded binary digits. Shannon's entropy measures the information contained in a message as opposed to the portion of the message that is determined (or predictable).
- The statistical entropy perspective was introduced in 1870 by Austrian physicist Ludwig Boltzmann, who established a new field of physics that provided the descriptive linkage between the macroscopic observation of nature and the microscopic view based on the rigorous treatment of a large ensembles of microstates that constitute thermodynamic systems.
- Ludwig Boltzmann defined entropy as a measure of the number of possible microscopic states (microstates) of a system in thermodynamic equilibrium, consistent with its macroscopic thermodynamic properties, which constitute the macrostate of the system.
- Entropy is a thermodynamic property just the same as pressure, volume, or temperature. Therefore, it connects the microscopic and the macroscopic world view. Boltzmann's principle is regarded as the foundation of statistical mechanics.

10.8 KEY WORDS

- **Hartley information:** In 1928, Hartley presented a simple measure of information. The term Hartley is named after Ralph Hartley, who suggested in 1928 to measure information using a logarithmic base equal to the number of distinguishable states in its representation, which would be the base 10 for a decimal digit.
- **Shannon entropy:** Shannon's entropy measures the information contained in a message as opposed to the portion of the message that is determined (or predictable).
- **Boltzmann entropy:** Ludwig Boltzmann defined entropy as a measure of the number of possible microscopic states (microstates) of a system in thermodynamic equilibrium, consistent with its macroscopic thermodynamic properties, which constitute the macrostate of the system.

10.9 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Define the classical measures of uncertainty.
2. Interpret the Hartley information.
3. State the Shannon entropy.
4. Explain the Boltzmann entropy.

Long-Answer Questions

1. Briefly define the classical measures of uncertainty.
2. Describe the Hartley information.
3. Explain the Shannon entropy. Give appropriate examples.
4. Analyse the Boltzmann entropy.

10.10 FURTHER READINGS

Klir, George J. and Bo Yuan. 1995. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. New Jersey: Prentice Hall Inc.

Klir, George J. and Tina A. Folger. 2009. *Fuzzy Sets, Uncertainty and Information*. New Delhi: Prentice Hall of India.

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UNIT 11 MEASURES OF DISSONANCE

NOTES

Structure

11.0 INTRODUCTION

Dissonance is the perception of contradictory information. Relevant items of information include a person's actions, feelings, ideas, beliefs, and values, and things in the environment. Dissonance is typically experienced as psychological stress when they participate in an action that goes against one or more of them. According to this theory, when two actions or ideas are not psychologically consistent with each other, people do all in their power to change them until they become consistent. The discomfort is triggered by the person's belief clashing with new information perceived, wherein they try to find a way to resolve the contradiction to reduce their discomfort.

In A Theory of Cognitive Dissonance (1957), Leon Festinger proposed that human beings strive for internal psychological consistency to function mentally in the real world. A person who experiences internal inconsistency tends to become psychologically uncomfortable and is motivated to reduce the cognitive dissonance. They tend to make changes to justify the stressful behaviour, either by adding new parts to the cognition causing the psychological dissonance (rationalization) or by avoiding circumstances and contradictory information likely to increase the magnitude of the cognitive dissonance (confirmation bias).

There are four theoretic paradigms of cognitive dissonance, the mental stress people suffer when exposed to information that is inconsistent with their beliefs, ideals or values: Belief Disconfirmation, Induced Compliance, Free Choice, and Effort Justification, which respectively explain what happens after a person acts inconsistently, relative to their intellectual perspectives; what happens after a person makes decisions and what are the effects upon a person who has expended much effort to achieve a goal. Common to each paradigm of cognitive-dissonance theory is the tenet: People invested in a given perspective shall—when confronted with contrary evidence—expend great effort to justify retaining the challenged perspective.

In this unit, you will study about the measures of dissonance, the body of evidence, consonant body of evidence, and related problems.

11.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the measures of dissonance
- Elaborate on the body of evidence
- Define consonant body of evidence

11.2 MEASURES OF DISSONANCE

Dissonance is the perception of contradictory information. Relevant items of information include a person's actions, feelings, ideas, beliefs, and values, and things in the environment. Dissonance is typically experienced as psychological stress when they participate in an action that goes against one or more of them. According to this theory, when two actions or ideas are not psychologically consistent with each other, people do all in their power to change them until they become consistent. The discomfort is triggered by the person's belief clashing with new information perceived, wherein they try to find a way to resolve the contradiction to reduce their discomfort.

In "*A Theory of Cognitive Dissonance* (1957)", Leon Festinger proposed that human beings strive for internal psychological consistency to function mentally in the real world. A person who experiences internal inconsistency tends to become psychologically uncomfortable and is motivated to reduce the cognitive dissonance. They tend to make changes to justify the stressful behaviour, either by adding new parts to the cognition causing the psychological dissonance (rationalization) or by avoiding circumstances and contradictory information likely to increase the magnitude of the dissonance (confirmation bias).

The term 'Measure of Dissonance' refers to the determination of the level of discomfort caused to the person. This can be caused by the relationship between two differing internal beliefs, or an action that is incompatible with the beliefs of the person. Two factors determine the degree of psychological dissonance caused by two conflicting cognitions or by two conflicting actions:

1. The importance of cognitions: the greater the personal value of the elements, the greater the magnitude of the dissonance in the relation. When the value of the importance of the two dissonant items is high, it is difficult to determine which action or thought is correct. Both have had

a place of truth, at least subjectively, in the mind of the person. Therefore, when the ideals or actions now clash, it is difficult for the individual to decide which takes priority.

2. Ratio of cognitions: the proportion of dissonant-to-consonant elements. There is a level of discomfort within each person that is acceptable for living. When a person is within that comfort level, the dissonant factors do not interfere with functioning. However, when dissonant factors are abundant and not enough in line with each other, one goes through a process to regulate and bring the ratio back to an acceptable level. Once a subject chooses to keep one of the dissonant factors, they quickly forget the other to restore peace of mind.

There is always some degree of dissonance within a person as they go about making decisions, due to the changing quantity and quality of knowledge and wisdom that they gain. The magnitude itself is a subjective measurement since the reports are self-relayed, and there is no objective way as yet to get a clear measurement of the level of discomfort.

Coping with the nuances of contradictory ideas or experiences is mentally stressful. It requires energy and effort to sit with those seemingly opposite things that all seem true. Festinger argued that some people would inevitably resolve dissonance by blindly believing whatever they wanted to believe.

Two cognitions or actions inconsistent with each other (e.g. not wanting to become drunk when out, but then drinking more wine) known as the dissonant relationship. Cognitive dissonance theory proposes that people seek psychological consistency between their expectations of life and the existential reality of the world. To function by that expectation of existential consistency, people continually reduce their cognitive dissonance in order to align their cognitions (perceptions of the world) with their actions.

Based on a brief overview of models and theories related to cognitive consistency from many different scientific fields, such as social psychology, perception, neurocognition, learning, motor control, system control, ethology, and stress, it has even been proposed that “All behaviour involving cognitive processing is caused by the activation of inconsistent cognitions and functions to increase perceived consistency”; that is, all behaviour functions to reduce cognitive inconsistency at some level of information processing. Indeed, the involvement of cognitive inconsistency has long been suggested for behaviours related to for instance curiosity, and aggression and fear, while it has also been suggested that the inability to satisfactorily reduce cognitive inconsistency may - dependent on the type and size of the inconsistency - result in stress.

Dissonance plays an important role in persuasion. To persuade people, you must cause them to experience dissonance, and then offer your proposal as a way to resolve the discomfort.

NOTES

11.3 BODY OF EVIDENCE

The mathematical theory of evidence has been introduced by Glenn Shafer in 1976 as a new approach to the representation of uncertainty. This theory can be represented under several distinct but more or less equivalent forms. The mathematical theory of evidence can be founded axiomatically on the notion of belief functions or on the allocation of belief masses to subsets of a frame of discernment.

Evidence or Dempster-Shafer Theory (DST) is used to model information which is both uncertain and imprecise. Such a piece of information can be apprehended by the mathematical model.

The conflict in **Fuzzy Evidence Theory (FET)** can be defined using the discounting factor which is determined by means of the reliability of corresponding **body of evidence**. The consistency of a body of evidence is not only related to the conflict with others, but also the uncertainty of itself. The conflict reflects the consistency between the body of evidence and others.

Taking the assumption that the truth lies in the majority, we can state that the smaller the degree of the conflict, the more consistent and reliable is the body of evidence. The uncertainty mirrors the decision precision of the body of evidence itself. The smaller the degree of uncertainty, the more supportive it is in the decision-making of the evidence. Both conflict degree and the uncertainty degree differ gradually.

Conflict Measure Based on Evidence Distance

In evidence theory, conflicting coefficient k is the most generally applied method for measuring the conflict between the bodies of evidence. However, in some conditions, the conflicting coefficient k cannot appropriately reflect the conflict between the bodies of evidence. To elucidate this problem, some new conflict measures are identified by mathematicians, such as evidence distance (Jousselme *et al*. 2001), pignistic probability distance (Smets 2005), singular value (Ke *et al*. 2013), correlation coefficient (Song *et al*. 2014) and belief entropy (Liu *et al*. 2019). In addition, some mathematicians recommend joining conflicting coefficient k with other conflict measures for representing the conflict between the bodies of evidence (Jiang *et al*. 2010; Li *et al*. 2016b).

Amongst these new methods for measuring conflict, the evidence distance method is the most extensively used method in the real applications. Evidence distance method was originally given by Jousselme, it has some good properties, such as satisfying three elements of distance and varying from 0 to 1.

The conflict degree between bodies of evidence influences the reasonability of the combination results and the uncertainty degree of the body of evidence influences the discriminability of the evidence combination results. Thus, conflict

NOTES

between bodies of evidence is the dominant factor to determine the discounting factors. When a body of evidence has a big conflict with others, it has low reliability and should be discounted. On the contrary, when a body of evidence has a small conflict with others, it has high reliability and should be maintained. However, when a body of evidence has middle conflict with others, the lower the uncertainty degree of itself is, the more discriminable for making a decision the combination results are, i.e., when two bodies of evidence have the equal conflict with others, the one with low degree of uncertainty of itself is more reliable than the one with high degree of uncertainty of itself.

11.4 CONSONANT BODY OF EVIDENCE

When the experimental data are represented by intervals and form a **consonant** or **combined (consonant-dissonant) body of evidence**, then their distribution is 'Fuzzy' which is characterised by means of overlaps. Additionally, when the experimental data are acquired and analysed using simultaneously independent estimations and proficient data, then the nature of the data is considered as combined, i.e., along with the probabilistic-statistical uncertainty then there exits possibility uncertainty. Consequently, in such situation the satisfactory outcomes can only be acquired when it is possible to do probabilistic-statistical analysis, in which a significant role belongs to fuzzy statistics used in decision-making system.

A random set is consonant if the focal sets can be arranged into a nested family of sets, that is, if $A_1 \subseteq A_2 \subseteq ... \subseteq A_n$. A belief (Bel) function derives from a consonant random set if and only if,

$$\mathbf{Bel}(\boldsymbol{A} \cap \boldsymbol{B}) = \mathbf{min}[\mathbf{Bel}(\boldsymbol{A}), \mathbf{Bel}(\boldsymbol{B})] \qquad \forall \boldsymbol{A}, \boldsymbol{B}$$

Or equivalently,

$$\mathbf{Pl}(\boldsymbol{A} \cup \boldsymbol{B}) = \mathbf{max}[\mathbf{Pl}(\boldsymbol{A}), \mathbf{Pl}(\boldsymbol{B})] \qquad \forall \boldsymbol{A}, \boldsymbol{B}$$

A **Belief measure** is a type of fuzzy measure; it has a dual measure, **Plausibility**. Under certain conditions Belief/Plausibility both become Probability; under certain different conditions Belief/Plausibility become Necessity/Possibility, respectively

Consonant random set-based plausibility functions or consonant plausibility functions were independently defined by Zadeh as possibility measures, in connection with the fuzzy sets. The consonant belief functions are referred as necessity measures.

The different fuzzy measures can be explained using set theory. The power set of a set is defined as the set of all subsets of that set including the null or empty set; for example, for the set {a, b, c} the power set is,

{ null, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c} }

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Each element of the power set that has a single element is called a singleton; {a}, {b}, and {c} are the singletons of the power set of {a, b, c}.

The set containing all the elements that are possible for the problem of interest is called the **Universe of Discourse (UOD)**.

A set cannot contain duplicate elements and the order of the elements does not matter. An event is defined as a collection of outcomes.

A fuzzy measure involves non-specificity when a degree of evidence is associated with a crisp subset that is not a singleton; that is, when a degree of evidence is applied to a crisp subset that has more than one outcome there is non-specificity.

Discord is associated with **conflict in evidence**; the greater the difference in the elements (outcomes) among the crisp sets to which evidence is given, the greater the conflict in the evidence.

Check Your Progress

1. Interpret the measures of dissonance.
2. What do you understand by the body of evidence?
3. Explain the consonant body of evidence.

11.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. The term 'Measure of Dissonance' refers to the determination of the level of discomfort caused to the person. This can be caused by the relationship between two differing internal beliefs, or an action that is incompatible with the beliefs of the person.
2. The mathematical theory of evidence has been introduced by Glenn Shafer in 1976 as a new approach to the representation of uncertainty. This theory can be represented under several distinct but more or less equivalent forms. The mathematical theory of evidence can be founded axiomatically on the notion of belief functions or on the allocation of belief masses to subsets of a frame of discernment.
3. When the experimental data are represented by intervals and form a consonant or combined (consonant-dissonant) body of evidence, then their distribution is 'Fuzzy' which is characterised by means of overlaps.

11.6 SUMMARY

NOTES

- Dissonance is the perception of contradictory information. Relevant items of information include a person's actions, feelings, ideas, beliefs, and values, and things in the environment.
- In A Theory of Cognitive Dissonance (1957), Leon Festinger proposed that human beings strive for internal psychological consistency to function mentally in the real world. A person who experiences internal inconsistency tends to become psychologically uncomfortable and is motivated to reduce the cognitive dissonance.
- The term 'Measure of Dissonance' refers to the determination of the level of discomfort caused to the person. This can be caused by the relationship between two differing internal beliefs, or an action that is incompatible with the beliefs of the person.
- Dissonance plays an important role in persuasion. To persuade people, you must cause them to experience dissonance, and then offer your proposal as a way to resolve the discomfort.
- The mathematical theory of evidence has been introduced by Glenn Shafer in 1976 as a new approach to the representation of uncertainty. This theory can be represented under several distinct but more or less equivalent forms.
- The mathematical theory of evidence can be founded axiomatically on the notion of belief functions or on the allocation of belief masses to subsets of a frame of discernment.
- When the experimental data are represented by intervals and form a consonant or combined (consonant-dissonant) body of evidence, then their distribution is 'Fuzzy' which is characterised by means of overlaps.

11.7 KEY WORDS

- **Measures of dissonance:** The term 'Measure of Dissonance' refers to the determination of the level of discomfort caused to the person. This can be caused by the relationship between two differing internal beliefs, or an action that is incompatible with the beliefs of the person.
- **Body of evidence:** The mathematical theory of evidence has been introduced by Glenn Shafer in 1976 as a new approach to the representation of uncertainty. This theory can be represented under several distinct but more or less equivalent forms.

- **Consonant body of evidence:** When the experimental data are represented by intervals and form a consonant or combined (consonant-dissonant) body of evidence, then their distribution is 'Fuzzy' which is characterised by means of overlaps.

NOTES

11.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Illustrate the measures of dissonance.
2. Elaborate on the body of evidence.
3. State the consonant body of evidence.

Long-Answer Questions

1. Describe briefly the measures of dissonance.
2. Explain the body of evidence with the help of example.
3. Analyse the consonant body of evidence.

11.9 FURTHER READINGS

Klir, George J. and Bo Yuan. 1995. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. New Jersey: Prentice Hall Inc.

Klir, George J. and Tina A. Folger. 2009. *Fuzzy Sets, Uncertainty and Information*. New Delhi: Prentice Hall of India.

Zimmermann, Hans-Jürgen. 1991. *Fuzzy Set Theory—And Its Applications*. New Delhi: Allied Publishers Private Limited.

BLOCK - IV
MEASURES OF CONFUSION, UNCERTAINTY AND INFORMATION

NOTES

UNIT 12 MEASURES OF CONFUSION

Structure

12.0 INTRODUCTION

Confusion is the quality or state of being bewildered or unclear. The term 'Acute Mental Confusion' is often used interchangeably with delirium in the International Statistical Classification of Diseases and Related Health Problems and the Medical Subject Headings publications to describe the pathology. These refer to the loss of orientation, or the ability to place oneself correctly in the world by time, location and personal identity. Mental confusion is sometimes accompanied by disordered consciousness (the loss of linear thinking) and memory loss (the inability to correctly recall previous events or learn new material).

Confusion may result from drug side effects or from a relatively sudden brain dysfunction. Acute confusion is often called delirium (or 'Acute Confusional State'), although delirium often includes a much broader array of disorders than simple confusion. These disorders include the inability to focus attention; various impairments in awareness, and temporal or spatial dis-orientation. Mental confusion can result from chronic organic brain pathologies, such as dementia.

The Confusion Assessment Method (CAM) is a diagnostic tool developed to allow non-psychiatric physicians and nurses to identify delirium in the healthcare setting. It was designed to be brief (less than 5 minutes to perform) and based on criteria from the Diagnostic and Statistical Manual of Mental Disorders (DSM). It includes four diagnostic criteria from the third edition of DSM (DSM-III-R): acute onset and fluctuating course, inattention, disorganized thinking, and altered level of consciousness. The CAM requires that a short cognitive test and interview is performed before it is completed.

Entropy is one of several ways to measure diversity. Specifically, Shannon entropy is the logarithm of ${}^1D$, the true diversity index with parameter equal to 1. The entropy of a given probability distribution of messages or symbols.

Entropy is often roughly used as a measure of the unpredictability of a cryptographic key, though its real uncertainty is unmeasurable. For example, a 128-bit key that is uniformly and randomly generated has 128 bits of entropy. It also takes (on average) $\mathbf{2^{127}}$ guesses to break by brute force. Entropy fails to capture the number of guesses required if the possible keys are not chosen uniformly. Instead, a measure called guesswork can be used to measure the effort required for a brute force attack.

In this unit, you will study about the measures of confusion, entropy like measures, and related problems.

12.1 OBJECTIVES

After going through this unit, you will be able to:

- Explain the measures of confusion
- Define the entropy like measures

12.2 MEASURES OF CONFUSION

The Confusion Assessment Method (CAM) is a diagnostic tool developed to allow non-psychiatric physicians and nurses to identify delirium in the healthcare setting. It was designed to be brief (less than 5 minutes to perform) and based on criteria from the Diagnostic and Statistical Manual of Mental Disorders (DSM). It includes four diagnostic criteria from the third edition of DSM (DSM-III-R): acute onset and fluctuating course, inattention, disorganized thinking, and altered level of consciousness. The CAM requires that a short cognitive test and interview is performed before it is completed.

The CAM has gained widespread use since its development in 1990. The tool was constructed through literature review and expert opinion and validated by psychiatrists using the DSM-III-R criteria for delirium. It has been translated into more than 20 languages and adapted for use in multiple other settings, including the ICU (CAM-ICU), nursing homes (NH-CAM), and Emergency department (B-CAM). The CAM has also been used in research settings. It has been found to predict poor outcomes in patients.

The CAM consists of two parts. The first tests for overall cognitive impairment relating to 9 features of delirium based on the DSM: acute onset, disorganized thinking, inattention, altered level of consciousness, disorientation, memory

NOTES

impairment, perceptual disturbances, psychomotor agitation or retardation, and altered sleep-wake cycle. The second part is used for the diagnosis and is made up of the four features best able to identify delirium.

A positive CAM was reported in the original validation study to have a sensitivity and specificity of 94–100% and 90–95% respectively for the presence of delirium. Some studies have reported lower sensitivities for the test in clinical use and when used by nurses. It is recommended that clinician judgement be applied as well when using the CAM to screen for delirium. While it may be used to assess for the presence of delirium, it is not useful to determine the severity of delirium or to monitor for clinical improvement or worsening.

In the field of machine learning and specifically the problem of statistical classification, a confusion matrix, also known as an error matrix, is a specific table layout that allows visualization of the performance of an algorithm, typically a supervised learning one (in unsupervised learning it is usually called a matching matrix). Each row of the matrix represents the instances in an actual class while each column represents the instances in a predicted class, or vice versa – both variants are found in the literature. The name stems from the fact that it makes it easy to see whether the system is confusing two classes (i.e. commonly mislabelling one as another).

Given a sample of 12 pictures, 8 of cats and 4 of dogs, where cats belong to class 1 and dogs belong to class 0,

Actual = [1,1,1,1,1,1,1,1,0,0,0,0],

Assume that a classifier that distinguishes between cats and dogs is trained, and we take the 12 pictures and run them through the classifier, and the classifier makes 9 accurate predictions and misses 3: 2 cats wrongly predicted as dogs (first 2 predictions) and 1 dog wrongly predicted as a cat (last prediction).

Prediction = [0,0,1,1,1,1,1,1,0,0,0,1],

With these two labelled sets (actual and predictions) we can create a confusion matrix that will summarize the results of testing the classifier:

Predicted class / Actual class	Cat	Dog
Cat	**6**	2
Dog	1	**3**

In this confusion matrix, of the 8 cat pictures, the system judged that 2 were dogs, and of the 4 dog pictures, it predicted that 1 were cats. All correct predictions are located in the diagonal of the table (highlighted in bold), so it is easy to visually inspect the table for prediction errors, as they will be represented by values outside the diagonal.

NOTES

In terms of sensitivity and specificity, the confusion matrix is as follows:

Actual class \ Predicted class	P	N
P	TP	FN
N	FP	TN

The confusion matrices discussed above have only two conditions: positive and negative. In some fields, confusion matrices can have more categories. For example, the table below summarises communication of a whistled language between two speakers, zero values omitted for clarity.

Vowel produced \ Perceived vowel	i	e	a	o	u
i	15		1		
e	1		1		
a			79	5	
o			4	15	3
u				2	2

In Shannon's original definitions, confusion refers to making the relationship between the cipher text and the symmetric key as complex and involved as possible; diffusion refers to dissipating the statistical structure of plaintext over the bulk of cipher text. This complexity is generally implemented through a well-defined and repeatable series of substitutions and permutations. Substitution refers to the replacement of certain components (usually bits) with other components, following certain rules. Permutation refers to manipulation of the order of bits according to some algorithm. To be effective, any non-uniformity of plaintext bits needs to be redistributed across much larger structures in the cipher text, making that non-uniformity much harder to detect.

In particular, for a randomly chosen input, if one flips the i-th bit, then the probability that the j-th output bit will change should be one half, for any i and j—this is termed the strict avalanche criterion. More generally, one may require that flipping a fixed set of bits should change each output bit with probability one half.

One aim of confusion is to make it very hard to find the key even if one has a large number of plaintext-cipher text pairs produced with the same key. Therefore, each bit of the cipher text should depend on the entire key, and in different ways on different bits of the key. In particular, changing one bit of the key should change the cipher text completely. The simplest way to achieve both diffusion and confusion is to use a substitution–permutation network. In these systems, the plaintext and the key often have a very similar role in producing the output, hence the same mechanism ensures both diffusion and confusion.

NOTES

12.3 ENTROPY LIKE MEASURES

In general, entropy permits us to make accurate statements and perform computations with favour to one of life's most pressing issues: not knowing how things will turn out. In other words, entropy is a measure of uncertainty. Entropy plays an important role in the analysis of the concept of information. Since, information theory was considered in the work of Shannon. There were some valid efforts to find more general measure of information, but their mostly results were formal, theoretical, and neither has provided better perception of the character of information.

Entropy measures are extensively useful to enumerate the complexity of dynamical systems in diverse fields. However, the practical approach of entropy methods is very challenging, due to the variety of entropy measures.

The entropy measurement can be done by the use of the following equation:

$$\Delta S = q/T \tag{12.1}$$

Where S represents the entropy, ΔS represents the change in entropy, q represents heat transfer, and T is the temperature. Using this equation it is possible to measure entropy changes using a calorimeter. The units of entropy is J/K.

The Shannon entropy is restricted to random variables taking discrete values. The corresponding formula for a continuous random variable with probability density function $f(x)$ with finite or infinite support $\mathbb{X}$ on the real line is defined by analogy, using the above form of the entropy as an expectation:

$$h[f] = \mathrm{E}[-\ln(f(x))] = -\int_{\mathbb{X}} f(x)\ln(f(x))\,dx.$$

This is the differential entropy (or continuous entropy). A precursor of the continuous entropy $h[f]$ is the expression for the functional — in the H-theorem of Boltzmann.

Although, the analogy between both functions is suggestive, the following question must be set: is the differential entropy a valid extension of the Shannon discrete entropy? Differential entropy lacks a number of properties that the Shannon discrete entropy has – it can even be negative – and corrections have been suggested, notably limiting density of discrete points.

To answer this question, a connection must be established between the two functions:

In order to obtain a generally finite measure as the bin size goes to zero. In the discrete case, the bin size is the (implicit) width of each of the n (finite or infinite) bins whose probabilities are denoted by p_n. As the continuous domain is generalised, the width must be made explicit.

To do this, start with a continuous function f discretized into bins of size Δ. By the mean-value theorem there exists a value xi in each bin such that

$$f(x_i)\Delta = \int_{i\Delta}^{(i+1)\Delta} f(x)\,dx$$

The integral of the function f can be approximated (in the Riemannian sense) by

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\Delta\to 0} \sum_{i=-\infty}^{\infty} f(x_i)\Delta$$

Where this limit and "bin size goes to zero" are equivalent.

We will denote

$$\mathrm{H}^{\Delta} := -\sum_{i=-\infty}^{\infty} f(x_i)\Delta \log(f(x_i)\Delta)$$

And expanding the logarithm, we have

$$\mathrm{H}^{\Delta} = -\sum_{i=-\infty}^{\infty} f(x_i)\Delta \log(f(x_i)) - \sum_{i=-\infty}^{\infty} f(x_i)\Delta \log(\Delta).$$

As $\Delta \to 0$, we have

$$\sum_{i=-\infty}^{\infty} f(x_i)\Delta \to \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\sum_{i=-\infty}^{\infty} f(x_i)\Delta \log(f(x_i)) \to \int_{-\infty}^{\infty} f(x)\log f(x)\,dx.$$

$\log(\Delta) \to \ -\infty$ as $\Delta \to 0$, requires a special definition of the differential or continuous entropy:

$$h[f] = \lim_{\Delta\to 0}\left(\mathrm{H}^{\Delta} + \log\Delta\right) = -\int_{-\infty}^{\infty} f(x)\log f(x)\,dx,$$

Which is, as said before, referred to as the differential entropy. This means that the differential entropy is not a limit of the Shannon entropy for $n \to \infty$. Rather, it differs from the limit of the Shannon entropy by an infinite offset (see also the article on information dimension).

NOTES

Check Your Progress

1. Define the term confusion.
2. Illustrate the measures of confusion.
3. Explain the entropy like measures.

12.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

- Confusion is the quality or state of being bewildered or unclear. The term 'Acute Mental Confusion' is often used interchangeably with delirium in the International Statistical Classification of Diseases and Related Health Problems and the Medical Subject Headings publications to describe the pathology.
- The Confusion Assessment Method (CAM) is a diagnostic tool developed to allow non-psychiatric physicians and nurses to identify delirium in the healthcare setting. It was designed to be brief (less than 5 minutes to perform) and based on criteria from the Diagnostic and Statistical Manual of Mental Disorders (DSM).
- Entropy measures are extensively useful to enumerate the complexity of dynamical systems in diverse fields. However, the practical approach of entropy methods is very challenging, due to the variety of entropy measures.

12.5 SUMMARY

- Confusion is the quality or state of being bewildered or unclear. The term 'Acute Mental Confusion' is often used interchangeably with delirium in the International Statistical Classification of Diseases and Related Health Problems and the Medical Subject Headings publications to describe the pathology.
- Confusion may result from drug side effects or from a relatively sudden brain dysfunction. Acute confusion is often called delirium (or 'Acute Confusional State'), although delirium often includes a much broader array of disorders than simple confusion.
- The Confusion Assessment Method (CAM) is a diagnostic tool developed to allow non-psychiatric physicians and nurses to identify delirium in the

NOTES

healthcare setting. It was designed to be brief (less than 5 minutes to perform) and based on criteria from the Diagnostic and Statistical Manual of Mental Disorders (DSM).

- A positive CAM was reported in the original validation study to have a sensitivity and specificity of 94–100% and 90–95% respectively for the presence of delirium. Some studies have reported lower sensitivities for the test in clinical use and when used by nurses.
- In the field of machine learning and specifically the problem of statistical classification, a confusion matrix, also known as an error matrix, is a specific table layout that allows visualization of the performance of an algorithm, typically a supervised learning one (in unsupervised learning it is usually called a matching matrix).
- The confusion matrices have only two conditions: positive and negative. In some fields, confusion matrices can have more categories.
- In Shannon's original definitions, confusion refers to making the relationship between the cipher text and the symmetric key as complex and involved as possible; diffusion refers to dissipating the statistical structure of plaintext over the bulk of cipher text.
- Entropy permits us to make accurate statements and perform computations with favour to one of life's most pressing issues: not knowing how things will turn out.
- In other words, entropy is a measure of uncertainty. Entropy plays an important role in the analysis of the concept of information. Since, information theory was considered in the work of Shannon.
- Entropy measures are extensively useful to enumerate the complexity of dynamical systems in diverse fields. However, the practical approach of entropy methods is very challenging, due to the variety of entropy measures.

12.6 KEY WORDS

- **Confusion:** Confusion is the quality or state of being bewildered or unclear. The term 'Acute Mental Confusion' is often used interchangeably with delirium in the International Statistical Classification of Diseases and Related Health Problems and the Medical Subject Headings publications to describe the pathology.
- **Confusion assessment method (CAM):** This is a diagnostic tool developed to allow non-psychiatric physicians and nurses to identify delirium in the healthcare setting. It was designed to be brief (less than 5 minutes to perform) and based on criteria from the Diagnostic and Statistical Manual of Mental Disorders (DSM).

NOTES

- **Entropy:** entropy is a measure of uncertainty. Entropy plays an important role in the analysis of the concept of information. Since, information theory was considered in the work of Shannon.
- **Entropy measures:** Entropy measures are extensively useful to enumerate the complexity of dynamical systems in diverse fields. However, the practical approach of entropy methods is very challenging, due to the variety of entropy measures.

12.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. State the term confusion.
2. Explain the measures of confusion.
3. Define the entropy like measures.

Long-Answer Questions

1. Discuss briefly the term confusion with the help of examples.
2. Explain the measures of confusion.
3. Analyse the entropy like measures. Give appropriate examples.

12.8 FURTHER READINGS

Klir, George J. and Bo Yuan. 1995. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. New Jersey: Prentice Hall Inc.

Klir, George J. and Tina A. Folger. 2009. *Fuzzy Sets, Uncertainty and Information*. New Delhi: Prentice Hall of India.

Zimmermann, Hans-Jürgen. 1991. *Fuzzy Set Theory—And Its Applications*. New Delhi: Allied Publishers Private Limited.

NOTES

UNIT 13 MEASURES OF NON-SPECIFICITY

Structure

13.0 INTRODUCTION

Non-specificity defines that two or more alternatives are left unspecified, by which we can represents a degree of imprecision. It only emphases on those focal elements which are with cardinality larger than 1. Non-specificity is a distinctive uncertainty type in the theory of belief functions when compared with the probability theory.

There are so many non-specificity measures one of them is the Hartley measure, which is originally for the classical set theory. In probability theory, there is only discord (or randomness or conflict). The mean of the belief intervals' lengths for all singletons is defined as the non-specificity.

Non-specificity is a characteristic of interval-valued fuzzy sets. Thus, non-specificity created by interval-valued fuzzy sets forms an essential linkage for the development of evidence theory with fuzzy sets. In belief functions, there are two types of uncertainty which are due to lack of knowledge: randomness and non-specificity.

Dempster–Shafer Theory (DST), is a general framework for reasoning with uncertainty, with understood connections to other frameworks such as probability, possibility and imprecise probability theories. First introduced by Arthur P. Dempster in the context of statistical inference, the theory was later developed by Glenn Shafer into a general framework for modeling epistemic uncertainty—a mathematical theory of evidence. The theory allows one to combine evidence from different sources and arrive at a degree of belief (represented by a mathematical object called belief function) that takes into account all the available evidence.

In this unit, you will study about the measures of non-specificity.

NOTES

13.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand what non-specificity is
- Explain the measures of non-specificity

13.2 MEASURES OF NON-SPECIFICITY

Non-specificity is a characteristic of interval-valued fuzzy sets. Thus, non-specificity created by interval-valued fuzzy sets forms an essential linkage for the development of evidence theory with fuzzy sets. In belief functions, there are two types of uncertainty which are due to lack of knowledge: randomness and non-specificity.

There are so many non-specificity measures one of them is the Hartley measure, which is originally for the classical set theory. In probability theory, there is only discord (or randomness or conflict). The mean of the belief intervals' lengths for all singletons is defined as the non-specificity.

Non-specificity defines that two or more alternatives are left unspecified, by which we can represents a degree of imprecision. It only emphases on those focal elements which are with cardinality larger than 1. Non-specificity is a distinctive uncertainty type in the theory of belief functions when compared with the probability theory.

Non-specificity is a type of Uncertainty with the reference of a set of possible alternatives. It is associated with cardinalities of relevant sets of alternatives. Suppose, A and B are two sets of alternatives, such that $A \subset B$, then the non-specificity of A is smaller than non-specificity of B. A set containing single element, has zero non-specificity.

Measures of non-specificity was discussed in detail by Dubois and Prade. Measure of non-specificity U, satisfies some axioms:

1. For all $A \in U(X)$, $U(A) \in [0, \infty)$,
2. $U(A) = 0$ iff there exists $x \in X$ such that $A = \{x\}$,
3. If $A \subset B$, then $U(A) \leq U(B)$.

Uncertainty measure in point 2 is obtained from the relative cardinality of fuzzy sets.

Theorem 13.1 $U(A) = -\log_2 \text{Card}(A)$.

Proof: By axiom 1,

If Card $(A) = 1 \Rightarrow U(A) = 0$ If Card $(A) = 1$ then $U(A) > 0$

Hence, $U(A) \in [0, \infty)$.

By axiom 2,

$U(A) = 0 \Leftrightarrow -\log_2 \text{Card}(A) = 0 \Leftrightarrow \text{Card}(A) = 1 \Leftrightarrow A = \{x\}$.

By axiom 3,

$A \subseteq B \Rightarrow \text{Card}(A) \leq \text{Card}(B)$

$\Rightarrow -\log_2 \text{Card}(A) \leq -\log_2 \text{Card}(B)$

$\Rightarrow U(A) \leq U(B)$

Hence proved.

Proposition:

$$D_R(A, B) = \frac{1}{n}\sum_i \left[\alpha_i \max\left\{0, \left|\left|\tilde{A}\right|_{\alpha_i} - \left|\tilde{B}\right|_{\alpha_i}\right|\right\}\right]$$

Proof: By axiom 2,

$$U(A) = 0 \Rightarrow \frac{1}{n D_R(A, B)}\sum_i \left[\alpha_i \max\left\{0, \left|\left|\tilde{A}\right|_{\alpha_i} - \left|\tilde{B}\right|_{\alpha_i}\right|\right\}\right] = 1$$

$$\Rightarrow D_R(A, B) = \frac{1}{n}\sum_i \left[\alpha_i \max\left\{0, \left|\left|\tilde{A}\right|_{\alpha_i} - \left|\tilde{B}\right|_{\alpha_i}\right|\right\}\right].$$

Proposition:

$U(A + B) < U(A) + U(B)$ if $U(A) = U(B)$

Proof:

$$U(A + B) = -\log_2\left(\frac{2}{n D_R(A, B)}\sum_i \left[\alpha_i \max\left\{0, \left|\left|\tilde{A}\right|_{\alpha_i} - \left|\tilde{B}\right|_{\alpha_i}\right|\right\}\right]\right)$$

$$< -2\log_2\left(\frac{}{n D_R(A, B)}\sum_i \left[\alpha_i \max\left\{0, \left|\left|\tilde{A}\right|_{\alpha_i} - \left|\tilde{B}\right|_{\alpha_i}\right|\right\}\right]\right)$$

$U(A + B) < U(A) + U(B)$.

Check Your Progress

1. What do you understand by the non-specificity?
2. Elaborate on the measures of non-specificity.
3. Prove that U (A + B) < U (A) + U (B) if U (A) = U (B).

13.3 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

NOTES

1. Non-specificity defines that two or more alternatives are left unspecified, by which we can represents a degree of imprecision. It only emphases on those focal elements which are with cardinality larger than 1. Non-specificity is a distinctive uncertainty type in the theory of belief functions when compared with the probability theory.
2. Measure of non-specificity U, satisfies some axioms:
   - For all $A \in U(X)$, $U(A) \in [0, \infty)$
   - $U(A) = 0$ iff there exists $x \in X$ such that $A = \{x\}$,
   - If $A \subset B$, then $U(A) \leq U(B)$.
3. $$U(A+B) = -\log_2\left(\frac{2}{nD_R(A,B)}\sum_i\left[\alpha_i \max\left\{0, \left|\left|\bar{A}\right|_{\alpha_i} - \left|\bar{B}\right|_{\alpha_i}\right|\right\}\right]\right)$$

$$< -2\log_2\left(\frac{}{nD_R(A,B)}\sum_i\left[\alpha_i \max\left\{0, \left|\left|\bar{A}\right|_{\alpha_i} - \left|\bar{B}\right|_{\alpha_i}\right|\right\}\right]\right)$$

$$\Rightarrow \quad U(A+B) < U(A) + U(B).$$

13.4 SUMMARY

- Non-specificity is a characteristic of interval-valued fuzzy sets. Thus, non-specificity created by interval-valued fuzzy sets forms an essential linkage for the development of evidence theory with fuzzy sets.
- In belief functions, there are two types of uncertainty which are due to lack of knowledge: randomness and non-specificity.
- There are so many non-specificity measures one of them is the Hartley measure, which is originally for the classical set theory. In probability theory, there is only discord (or randomness or conflict).
- Non-specificity is a distinctive uncertainty type in the theory of belief functions when compared with the probability theory.
- Non-specificity is a type of Uncertainty with the reference of a set of possible alternatives. It is associated with cardinalities of relevant sets of alternatives. Suppose, A and B are two sets of alternatives, such that $A \subset B$, then the non-specificity of A is smaller than non-specificity of B.

- A set containing single element, has zero non-specificity. Measures of non-specificity was discussed in detail by Dubois and Prade.

NOTES

13.5 KEY WORDS

- **Non-specificity:** Non-specificity is a characteristic of interval-valued fuzzy sets. Thus, non-specificity created by interval-valued fuzzy sets forms an essential linkage for the development of evidence theory with fuzzy sets.
- **Measure of non-specificity:** Measure of non-specificity U, satisfies some axioms:
  - For all $A \in U(X)$, $U(A) \in [0, \infty)$,
  - $U(A) = 0$ iff there exists $x \subset X$ such that $A = \{x\}$,
  - If $A \subset B$, then $U(A) \leq U(B)$.

13.6 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Elaborate on the non-specificity.
2. State the measures of non-specificity.
3. Prove that $U(A + B) < U(A) + U(B)$ if $U(A) = U(B)$.

Long-Answer Questions

1. Discuss briefly the non-specificity with the help of examples.
2. Explain the measures of non-specificity.
3. State and prove the theorem $U(A) = -\log_2 \text{Card}(A)$.

13.7 FURTHER READINGS

Klir, George J. and Bo Yuan. 1995. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. New Jersey: Prentice Hall Inc.

Klir, George J. and Tina A. Folger. 2009. *Fuzzy Sets, Uncertainty and Information*. New Delhi: Prentice Hall of India.

Zimmermann, Hans-Jürgen. 1991. *Fuzzy Set Theory—And Its Applications*. New Delhi: Allied Publishers Private Limited.

UNIT 14 UNCERTAIN INFORMATION

NOTES

Structure

14.0 INTRODUCTION

Uncertain information can be define as a piece of information which is valid under certain assumptions, but it is not altogether sure that these assumptions really hold. Variation in the assumptions describe the different information. Such uncertain piece of information, and assumption-based reasoning permits to deduce certain conclusions.

Uncertain information is found in abundance today on the web, in sensor networks, within enterprises both in their structured and unstructured sources. For example, there may be uncertainty regarding the address of a customer in an enterprise information set, or the temperature readings captured by a sensor due to aging of the sensor. In 2012 IBM called out managing uncertain information at scale in its global technology outlook report that presents a comprehensive analysis looking three to ten years into the future seeking to identify significant, disruptive technologies that will change the world.

In order to make confident business decisions based on real-world information, analyses must necessarily account for many different kinds of uncertainty present in very large amounts of information. Analyses based on uncertain information will have an effect on the quality of subsequent decisions, so the degree and types of inaccuracies in this uncertain information cannot be ignored.

Uncertain information is found in the area of sensor networks; text where noisy text is found in abundance on social media, web and within enterprises where the structured and unstructured information may be old, outdated, or plain incorrect; in modeling where the mathematical model may only be an approximation of the actual process. When representing such information in an information base, some indication of the probability of the correctness of the various values also needs to be estimated.

In this unit, you will study about the uncertain and information - syntactic, semantic, and pragmatic.

NOTES

14.1 OBJECTIVES

After going through this unit, you will be able to:

- Elaborate on the uncertain information
- Define the syntactic information
- Explain the semantic information
- Analyse the pragmatic information

14.2 UNCERTAIN INFORMATION

There are three main models of uncertain information in databases. In attribute uncertainty, each uncertain attribute in a tuple is subject to its own independent probability distribution. For example, if readings are taken of temperature and wind speed, each would be described by its own probability distribution, as knowing the reading for one measurement would not provide any information about the other.

In correlated uncertainty, multiple attributes may be described by a joint probability distribution. For example, if readings are taken of the position of an object, and the *x*- and *y*-coordinates stored, the probability of different values may depend on the distance from the recorded coordinates. As distance depends on both coordinates, it may be appropriate to use a joint distribution for these coordinates, as they are not independent.

In tuple uncertainty, all the attributes of a tuple are subject to a joint probability distribution. This covers the case of correlated uncertainty, but also includes the case where there is a probability of a tuple not belonging in the relevant relation, which is indicated by all the probabilities not summing to one. For example, assume we have the following tuple from a probabilistic database:

(a, 0.4) | (b, 0.5)

Then, the tuple has 10% chance of not existing in the database.

Uncertain information can be define as a piece of information which is valid under certain assumptions, but it is not altogether sure that these assumptions really hold. Variation in the assumptions describe the different information. Such uncertain piece of information, and assumption-based reasoning permits to deduce certain conclusions.

NOTES

Uncertain information is found in the area of sensor networks; text where noisy text is found in abundance on social media, web and within enterprises where the structured and unstructured information may be old, outdated, or plain incorrect; in modeling where the mathematical model may only be an approximation of the actual process. When representing such information in an information base, some indication of the probability of the correctness of the various values also needs to be estimated.

Uncertain information is found in abundance today on the web, in sensor networks, within enterprises both in their structured and unstructured sources. For example, there may be uncertainty regarding the address of a customer in an enterprise information set, or the temperature readings captured by a sensor due to aging of the sensor. In 2012 IBM called out managing uncertain information at scale in its global technology outlook report that presents a comprehensive analysis looking three to ten years into the future seeking to identify significant, disruptive technologies that will change the world.

Uncertainty analysis investigates the uncertainty of variables that are used in decision-making problems in which observations and models represent the knowledge base. In other words, uncertainty analysis aims to make a technical contribution to decision-making through the quantification of uncertainties in the relevant variables. A calibrated parameter does not necessarily represent reality, as reality is much more complex. Any prediction has its own complexities of reality that cannot be represented uniquely in the calibrated model; therefore, there is a potential error. Such error must be accounted for when making management decisions on the basis of model outcomes.

The uncertainty u can be expressed in a number of ways. It may be defined by the absolute error Δx. Uncertainties can also be defined by the relative error $(\Delta x)/x$, which is usually written as a percentage. Most commonly, the uncertainty on a quantity is quantified in terms of the standard deviation, σ, which is the positive square root of the variance. The value of a quantity and its error are then expressed as an interval $x \pm u$. If the statistical probability distribution of the variable is known or can be assumed, it is possible to derive confidence limits to describe the region within which the true value of the variable may be found. For example, the 68% confidence limits for a one-dimensional variable belonging to a normal distribution are approximately $\pm$ one standard deviation σ from the central value x, which means that the region $x \pm \sigma$ will cover the true value in roughly 68% of cases.

If the uncertainties are correlated then covariance must be taken into account. Correlation can arise from two different sources. First, the measurement errors may be correlated. Second, when the underlying values are correlated across a population, the uncertainties in the group averages will be correlated.

Random-Fuzzy Variable (RFV) is a type 2 fuzzy variable, defined using the mathematical possibility theory, and used to represent the entire information associated to a measurement result. It has an internal possibility distribution and an external possibility distribution called membership functions. The internal distribution is the uncertainty contributions due to the systematic uncertainty and the bounds of the RFV are because of the random contributions. The external distribution gives the uncertainty bounds from all contributions.

14.2.1 Syntactic Information

In logic, syntax is anything having to do with formal languages or formal systems without regard to any interpretation or meaning given to them. Syntax is concerned with the rules used for constructing, or transforming the symbols and words of a language, as contrasted with the semantics of a language which is concerned with its meaning.

Syntax is usually associated with the rules (or grammar) governing the composition of texts in a formal language that constitute the well-formed formulas of a formal system. The symbols, formulas, systems, theorems, proofs, and interpretations expressed in formal languages are syntactic entities whose properties may be studied without regard to any meaning they may be given, and, in fact, need not be given any.

In computer science, the term syntax refers to the rules governing the composition of well-formed expressions in a programming language. As in mathematical logic, it is independent of semantics and interpretation.

A formal system $\mathcal{S}$ is syntactically complete (also deductively complete, maximally complete, negation complete or simply complete) iff for each formula A of the language of the system either A or ¬A is a theorem of $\mathcal{S}$. In another sense, a formal system is syntactically complete iff no unprovable axiom can be added to it as an axiom without introducing an inconsistency. Truth-functional propositional logic and first-order predicate logic are semantically complete, but not syntactically complete (for example the propositional logic statement consisting of a single variable “a” is not a theorem, and neither is its negation, but these are not tautologies). Gödel’s incompleteness theorem shows that no recursive system that is sufficiently powerful, such as the Peano axioms, can be both consistent and complete.

14.2.2 Semantic Information

Semantic information states an information, which is in some sense, is meaningful for a system, rather than just correlational. It plays an important role in many fields like biology, cognitive science, artificial intelligence, information theory, and philosophy. Semantic information has been compared with syntactic information, it measures several statistical correlation between two systems, with no contemplation of what such correlations actually ‘Mean’.

In general, semantics is the study of meaning, reference, or truth. The term can be used to refer to subfields of several distinct disciplines, including philosophy, linguistics and computer science. Formal semantics seeks to identify domain-specific mental operations which speakers perform when they compute a sentence’s

NOTES

meaning on the basis of its syntactic structure. Theories of formal semantics are typically floated on top of theories of syntax such as generative syntax or combinatory categorial grammar and provide a model theory based on mathematical tools such as typed lambda calculi. The field's central ideas are rooted in early twentieth century philosophical logic as well as later ideas about linguistic syntax. It emerged as its own subfield in the 1970s after the pioneering work of Richard Montague and Barbara Partee and continues to be an active area of research.

In computer science, the term semantics refers to the meaning of language constructs, as opposed to their form (syntax). According to Euzenat, semantics "provides the rules for interpreting the syntax which do not provide the meaning directly but constrains the possible interpretations of what is declared".

Conceptual semantics refers to an effort to explain properties of argument structure. The assumption behind this theory is that syntactic properties of phrases reflect the meanings of the words that head them. With this theory, linguists can better deal with the fact that subtle differences in word meaning correlate with other differences in the syntactic structure that the word appears in. The way this is gone about is by looking at the internal structure of words. These small parts that make up the internal structure of words are termed semantic primitives.

Computational semantics is focused on the processing of linguistic meaning. In order to do this concrete algorithms and architectures are described. Within this framework the algorithms and architectures are also analysed in terms of decidability, time/space complexity, data structures that they require and communication protocols.

14.2.3 Pragmatic Information

The word pragmatics derives via Latin pragmaticus from the Greek (pragmatikós), meaning amongst others "Fit for Action", which comes from (prâgma) "Deed, Act", in turn from the verb "To do, To act, To pass over, To practise, To achieve".

The pragmatic information content is the information content received by a recipient; it is focused on the recipient and defined in contrast to Claude Shannon's information definition, which focuses on the message. The pragmatic information measures the information received, not the information contained in the message.

Pragmatic information theory requires not only a model of the sender and how it encodes information, but also a model of the receiver and how it acts on the information received. The determination of pragmatic information content is a precondition for the determination of the value of information. Pragmatics of communication is the observable effect a communication act (here receiving a message) has on the actions of the recipient. The pragmatic information content of a message may be different for different recipients or the same message may have the same content. Weizsäcker used the concept of novelty and irrelevance to separate information which is pragmatically useful or not.

NOTES

The pragmatic theory of information is derived from Charles Sanders Peirce's general theory of signs and inquiry. Peirce explored a number of ideas about information throughout his career. One set of ideas is about the "Laws of Information" having to do with the logical properties of information. Another set of ideas about "Time and Thought" have to do with the dynamic properties of inquiry. All of these ideas contribute to the pragmatic theory of inquiry. Peirce set forth many of these ideas very early in his career, periodically returning to them on scattered occasions until the end, and they appear to be implicit in much of his later work on the logic of science and the theory of signs, but he never developed their implications to the fullest extent. The 20th century thinker Ernst Ulrich and his wife Christine von Weizsäcker reviewed the pragmatics of information; their work is reviewed by Gennert.

Pragmatic rules are used quite frequently by speakers but are rarely noticed unless the unspoken rules of pragmatics are broken. The ability to understand another speaker's intended meaning is called pragmatic competence.

Check Your Progress

1. Define the uncertain information.
2. Elaborate on the syntactic information.
3. Illustrate the semantic information.
4. Explain the pragmatic information.

14.3 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Uncertain information can be define as a piece of information which is valid under certain assumptions, but it is not altogether sure that these assumptions really hold. Variation in the assumptions describe the different information. Such uncertain piece of information, and assumption-based reasoning permits to deduce certain conclusions.
2. In logic, syntax is anything having to do with formal languages or formal systems without regard to any interpretation or meaning given to them. Syntax is concerned with the rules used for constructing, or transforming the symbols and words of a language, as contrasted with the semantics of a language which is concerned with its meaning.
3. Semantic information states an information, which is in some sense, is meaningful for a system, rather than just correlational. Semantic information has been compared with syntactic information, it measures several statistical correlation between two systems, with no contemplation of what such correlations actually 'Mean'.

NOTES

4. The pragmatic information content is the information content received by a recipient; it is focused on the recipient and defined in contrast to Claude Shannon's information definition, which focuses on the message. The pragmatic information measures the information received, not the information contained in the message.

14.4 SUMMARY

- Uncertain information can be define as a piece of information which is valid under certain assumptions, but it is not altogether sure that these assumptions really hold.
- There are three main models of uncertain information in databases. In attribute uncertainty, each uncertain attribute in a tuple is subject to its own independent probability distribution.
- Uncertain information can be define as a piece of information which is valid under certain assumptions, but it is not altogether sure that these assumptions really hold.
- In logic, syntax is anything having to do with formal languages or formal systems without regard to any interpretation or meaning given to them. Syntax is concerned with the rules used for constructing, or transforming the symbols and words of a language.
- In computer science, the term syntax refers to the rules governing the composition of well-formed expressions in a programming language. As in mathematical logic, it is independent of semantics and interpretation.
- Semantic information states an information, which is in some sense, is meaningful for a system, rather than just correlational. It plays an important role in many fields like biology, cognitive science, artificial intelligence, information theory, and philosophy.
- In computer science, the term semantics refers to the meaning of language constructs, as opposed to their form (syntax). According to Euzenat, semantics "provides the rules for interpreting the syntax which do not provide the meaning directly but constrains the possible interpretations of what is declared".
- The pragmatic information content is the information content received by a recipient; it is focused on the recipient and defined in contrast to Claude Shannon's information definition, which focuses on the message.
- The pragmatic information measures the information received, not the information contained in the message.
- The pragmatic information content of a message may be different for different recipients or the same message may have the same content. Weizsäcker

used the concept of novelty and irrelevance to separate information which is pragmatically useful or not.

- Pragmatic rules are used quite frequently by speakers but are rarely noticed unless the unspoken rules of pragmatics are broken. The ability to understand another speaker's intended meaning is called pragmatic competence.

NOTES

14.5 KEY WORDS

- **Uncertain information:** Uncertain information can be define as a piece of information which is valid under certain assumptions, but it is not altogether sure that these assumptions really hold.
- **Syntactic information:** In logic, syntax is anything having to do with formal languages or formal systems without regard to any interpretation or meaning given to them.
- **Semantic information:** Semantic information states an information, which is in some sense, is meaningful for a system, rather than just correlational. It plays an important role in many fields like biology, cognitive science, artificial intelligence, information theory, and philosophy.
- **Pragmatic information:** The pragmatic information content of a message may be different for different recipients or the same message may have the same content. Weizsäcker used the concept of novelty and irrelevance to separate information which is pragmatically useful or not.

14.6 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. State the uncertain information.
2. Interpret the syntactic information.
3. Define the semantic information.
4. Elaborate on the pragmatic information.

Long-Answer Questions

1. Briefly discuss the uncertain information.
2. Describe the syntactic information. Give appropriate examples.
3. Explain the semantic information.
4. Analyse the pragmatic information.

14.7 FURTHER READINGS

NOTES

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